

Isotropic Ornstein-Uhlenbeck Flows

by

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Summary. Isotropic Brownian flows (IBFs) are a fairly natural class of stochastic flows which has been studied extensively by various authors. Their rich structure allows for explicit calculations in several situations and makes them a natural object to start with if one wants to study more general stochastic flows. Often the intuition gained by understanding the problem in the context of IBFs transfers to more general situations. However, the obvious link between stochastic flows, random dynamical systems and ergodic theory cannot be exploited in its full strength as the IBF does not have an invariant probability measure but rather an infinite one. Isotropic Ornstein-Uhlenbeck flows (IOUFs) are in a sense localized IBFs and do have an invariant probability measure. The imposed linear drift destroys the translation invariance of the IBF, but many other important structure properties like the Markov property of the distance process remain valid and allow for explicit calculations in certain situations. The fact that IOUFs have invariant probability measures allows one to apply techniques from random dynamical systems theory. We demonstrate this by applying the results of Ledrappier and Young to calculate the Hausdorff dimension of the statistical equilibrium of an IOUF.

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1 Introduction

Isotropic Brownian flows (IBFs) form a class of stochastic flows, which is intimately connected to the Lebesgue measure in view of the following facts:

1. The one-point motion of an IBF has the Lebesgue measure as its invariant measure (modulo multiplicative constant).
2. The distribution of an IBF is invariant under rigid transformations of \mathbb{R}^d , which also preserve, and in fact characterize, the Lebesgue measure in \mathbb{R}^d up to a multiplicative constant.

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The Gaussian measures are fairly common in probability theory and therefore it is natural to look at the family of stochastic flows which is connected to the family of centered Gaussian measures on $\mathcal{B}(\mathbb{R}^d)$ in the above sense. The one-point motion of such a flow should be an Ornstein-Uhlenbeck process, and its law must be invariant with respect to rotations of \mathbb{R}^d . The procedure of obtaining an Ornstein-Uhlenbeck process from a given Brownian motion, via an SDE with linear drift can also be performed for flows. Here is the definition of the isotropic Ornstein-Uhlenbeck flows, using this procedure:

Take a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $F(t, x, \omega)$ be an isotropic Brownian field with C^4 covariance tensor b , i.e. F generates an isotropic Brownian flow.

For the convenience of the reader we will first revise some important facts on Isotropic Brownian Flows and their generating fields. The \mathbb{R}^d -valued random vector field $\{F(t, x) : t \geq 0, x \in \mathbb{R}^d\}$ is an isotropic Brownian field if $(t, x) \mapsto F(t, x)$ is an almost surely continuous, time homogeneous, centered Gaussian process whose distribution is invariant with respect to rigid transformations (translation, reflection, rotation) of the spatial variable x . The covariance structure of F is given by $\text{cov}(F(s, x), F(t, y)) = (s \wedge t)b(x - y)$, where $b : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ is a so called isotropic covariance tensor. The isotropy of F (distributional invariance with respect to rotations and reflections in the spatial variable) implies that

$$b(x) = O^T b(Ox) O \quad (1.1)$$

for all O from the orthogonal group \mathcal{O}^d . Following [4] we will throughout assume that b has continuous and bounded derivatives of order up to 4. Some further details on the covariance tensor b will be given in the beginning of Section 2. An IBF is defined to be the flow generated by the Kunita-type SDE $\psi_{s,t}(x) = x + \int_s^t F(du, \psi_{s,u}(x))$. Let us note that the Proposition 2.2 is valid for IBFs if one puts $c = 0$ there and replaces the words *Ornstein-Uhlenbeck diffusion* by the words *Brownian Motion*.

For more details on isotropic Brownian flows and their generating fields consult e.g. [4] and [18].

For a $c > 0$ define the semimartingale field

$$V(t, x, \omega) = F(t, x, \omega) - \int_0^t c x ds = F(t, x, \omega) - cxt.$$

It is an isotropic Brownian field with a linear drift towards the origin and its local characteristic (b, l_c) (with $l_c(x) := cx$) belongs to the class $(B_{ub}^4, B_{ub}^\infty)$ (see [17] for a comprehensive study of semimartingales with spatial parameter). The obvious analogy with the characterization of an Ornstein-Uhlenbeck process as a Brownian motion in a quadratic potential motivates the following definition.

Definition 1.1. An *isotropic Ornstein-Uhlenbeck flow*, shortly *IOUF* is a stochastic flow of $C^{3,\delta}$ (for arbitrary $\delta > 0$) diffeomorphisms ϕ , generated by (via a Kunita-type SDE) a semimartingale field $V(t, x, \omega)$ as above, i.e. $\phi_{s,t}(x) = x + \int_s^t V(du, \phi_{s,u}(x))$.

Clearly, an IOUF is not translation invariant, but still retains most of the nice properties of an IBF, like the most central fact that the distance process is a diffusion. In contrast to IBFs, IOUFs have the very important feature that the one-point motion admits an invariant *probability* measure, and therefore one has at his disposal some powerful tools from random

dynamical systems theory. To be more specific the linearization of an IBF does not strictly fulfill the conditions of the multiplicative ergodic theorem as the flow does not have an invariant *probability* measure and therefore we only get some restricted statement instead (the Lyapunov exponents are still well defined). On the other hand the linearization of an IOUF satisfies the conditions of the multiplicative ergodic theorem and we have its statement in its full strength.

To an IOUF as well as to an IBF one can associate a statistical equilibrium, which is the proper extension of the notion of an invariant measure to stochastic flows. It is a random measure on \mathbb{R}^d (we consider \mathbb{R}^d -flows), which is stationary under the action of the flow, and moreover the pullback of the invariant measure for the one point motion converges to the statistical equilibrium (more on this in Section 3). For an IBF we have the following: In the volume preserving case the statistical equilibrium is the Lebesgue measure itself, and in the case of a negative top Lyapunov exponent Darling and LeJan ([10]) showed that the statistical equilibrium is a zero measure. In the rest of the cases the statistical equilibrium is a non-trivial random measure being a.s. singular to the Lebesgue measure. In the case of an IOUF the statistical equilibrium is a random probability measure being Dirac if the top Lyapunov exponent is strictly negative and a.s. diffuse if it is strictly positive. Ledrappier and Young were able to explicitly calculate (in terms of the Lyapunov exponents) the Hausdorff dimension of the statistical equilibrium for an RDS on a compact manifold ([22]) and moreover, to link it to the entropy of the system, thus giving a precise meaning of the intuitively clear interplay between the entropy and dimensionality of the invariant measure of a dynamical system ([20], [21], [19]). In Section 3 we apply their result to IOUFs with the help of a compactification argument, which works precisely because an IOUF has an invariant probability measure, and consequently breaks down for isotropic Brownian flows. We conjecture that the Hausdorff dimension of an IBF with strictly positive top Lyapunov exponent can be obtained as a limit of the dimensions of the corresponding dimensions for the IOUFs with $c \rightarrow 0$, however we have not been able to prove this yet. The conjecture is trivially true for the extreme situations of an IBF with negative top Lyapunov exponent and a volume preserving IBF.

Another motivation for considering the IOUF is the natural question of letting an isotropic Brownian flow evolve in a localizing potential, i.e. for some field $U: \mathbb{R}^d \rightarrow \mathbb{R}$ consider the IBF ϕ^U enforced by the potential U , that is, ϕ^U is generated via an SDE driven by the semimartingale field

$$V^U(t, x, \omega) = F(t, x, \omega) - \nabla U(x)t.$$

The IOUFs correspond to a quadratic potential $U = \frac{c}{2}|x|^2$. However, adding a drift typically will destroy many of the nice properties of an IBF. Clearly, the IOUFs are not translation invariant, but still retain most of the rich structure of the IBFs, like as already mentioned, the most central fact that the distance process is a diffusion. Moreover the IOUFs can be considered as localized IBFs as their one-point motions have an invariant *probability* measures. As we shall see, they are also localized as flows since they posses weak random attractors.

The paper is organized as follows: In the next section we provide some “infrastructure” facts on IOUFs. In Section 3 we introduce the notion of statistical equilibrium and give its Hausdorff dimension for an IOUF. In Section 4 we give a brief introduction to the notion of random attractors and then establish the existence of weak attractors for IOUFs.

2 General facts on IOUF

Let $(\Omega, \mathcal{F}, (\mathcal{F}_s^t)_{0 \leq s \leq t}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions i.e. every \mathcal{F}_s^t is complete and the filtration is right-continuous in s and t . Let ϕ be an IOUF, generated as above by the field $F(t, x, \omega) - cxt$.

Notation 2.1. We will write ϕ_t for $\phi_{0,t}$.

The distributions of F and ϕ are uniquely determined by the *covariance tensor*

$$b : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}, \quad (x, y) \mapsto \mathbb{E}F(1, x)F(1, y) \text{ and } c > 0.$$

According to Yaglom [26, Section 4] (see also [4]) the isotropy property (1.1) implies that b has necessarily the form

$$b_{i,j}(x) = \begin{cases} (B_L(|x|) - B_N(|x|)) \frac{x_i x_j}{|x|^2} + \delta_{i,j} B_N(|x|) & \text{for } x \neq 0 \\ \delta_{i,j} B_L(0) = \delta_{i,j} B_N(0) & \text{for } x = 0, \end{cases} \quad (2.2)$$

where B_L and B_N are the so called longitudinal and transversal correlation functions defined by

$$\begin{aligned} B_L(r) &= b_{p,p}(re_p), \quad r \geq 0 \quad \text{and} \\ B_N(r) &= b_{p,p}(re_q), \quad r \geq 0, \quad p \neq q. \end{aligned}$$

In the above formulas e_i denotes the i -th standard basis vector in \mathbb{R}^d . In particular, (1.1) implies $b(0) = k \cdot \text{Id}_{\mathbb{R}^d}$ for some positive constant k . Without loss of generality, as it is done in [4], we assume that $b(0) = \text{Id}_{\mathbb{R}^d}$. In this case the one-point-motion is a standard Brownian motion and we also get $B_L(0) = B_N(0) = 1$. The general case $b(0) = k \cdot \text{Id}_{\mathbb{R}^d}$ can be obtained via rescaling the time by the constant factor \sqrt{k} . B_L and B_N are bounded C^4 functions with bounded derivatives up to order four. Set

$$\begin{aligned} \beta_L &:= -\partial_p \partial_p b_{p,p}(0) = -B_L''(0) \\ \beta_N &:= -\partial_q \partial_q b_{p,p}(0) = -B_N''(0) \quad q \neq p, \end{aligned}$$

where $B_L''(0)$ and $B_N''(0)$ denote the right-hand second derivatives of B_L and B_N at zero. We refer the reader to [4] and [18] for more details.

Proposition 2.2. *Let ϕ be an isotropic Ornstein-Uhlenbeck flow. Then for $x, y \in \mathbb{R}^d$*

- (a) *ϕ is Brownian and isotropic (invariant with respect to orthogonal transformations).*
- (b) *The one-point motion $\{\phi_t(x) : t \in \mathbb{R}\}$ of ϕ is an Ornstein-Uhlenbeck diffusion with generator*

$$\mathcal{L} := -c \sum_{i=1}^d x_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} = -c \sum_{i=1}^d x_i \frac{\partial}{\partial x_i} + \frac{1}{2} \Delta.$$

(c) The difference process $\{\phi_t(x) - \phi_t(y) : t \in \mathbb{R}_+\}$ is a diffusion with generator

$$\mathcal{L}_d := -c \sum_{i=1}^d x_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^d (\delta_{ij} - b_{i,j}(x)) \frac{\partial^2}{\partial x_i \partial x_j}.$$

(d) The distance process $\{|\phi_t(x) - \phi_t(y)| : t \in \mathbb{R}_+\}$ is a diffusion with generator

$$\mathcal{A} = (1 - B_L(r)) \frac{d^2}{dr^2} + ((d-1) \frac{1 - B_N(r)}{r} - cr) \frac{d}{dr}.$$

Proof. (a) Recall, that a stochastic flow is called *Brownian* if it has independent increments (see [17], Section 4.1, page 116), i.e. for all $n \in \mathbb{N}$ and $0 \leq t_1 \leq t_2 \leq \dots \leq t_n$ the random variables $(\phi_{t_1, t_2}, \dots, \phi_{t_{n-1}, t_n})$ are independent. This property is obvious as the field F has independent increments.

The invariance with respect to orthogonal transformations is a direct consequence of the invariance of $F(t, x)$ and the drift with respect to these operations. **(b)** Using Definition 1.1 we get for the joint quadratic variation of the components of the one-point motion

$$\langle \phi^i(x), \phi^j(x) \rangle_t = \left\langle \int_0^t F^i(ds, \phi_s(x)), \int_0^t F^j(ds, \phi_s(x)) \right\rangle_t = b_{i,j}(0)t = \delta_{ij}t.$$

Let $f \in C_0^2(\mathbb{R}^d : \mathbb{R})$ be arbitrary. An application of Itô's formula gives

$$\begin{aligned} f(\phi_t(x)) - f(x) &= \sum_{i=1}^d \int_0^t \partial_i f(\phi_s(x)) d\phi_s^i(x) + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij}^2 f(\phi_s(x)) d\langle \phi^i(x), \phi^j(x) \rangle_s \\ &= \sum_{i=1}^d \int_0^t \partial_i f(\phi_s(x)) F^i(ds, \phi_s(x)) - c \sum_{i=1}^d \int_0^t \phi_s^i(x) \partial_i f(\phi_s(x)) ds + \frac{1}{2} \sum_{i,j=1}^d \delta_{ij} \int_0^t \partial_{ij}^2 f(\phi_s(x)) ds, \end{aligned}$$

that is $t \mapsto f(\phi_t(x)) - f(\phi_0(x)) - \int_0^t (\mathcal{L}f)(\phi_s(x)) ds$ is a continuous martingale for all $x \in \mathbb{R}^d$, and therefore $\phi_t(x)$ is a diffusion with generator \mathcal{L} (see [14], Propositions 4.6 and 4.11, Section 5.4.B).

(c) Let $x, y \in \mathbb{R}^d$, $x \neq y$ be arbitrary but fixed. We clearly have the equality:

$$\phi_t(x) - \phi_t(y) = x - y - c \int_0^t (\phi_t(x) - \phi_t(y)) ds + \int_0^t F(ds, \phi_s(x)) - \int_0^t F(ds, \phi_s(y)).$$

Observe the following computation for the joint quadratic variation of the difference process

$$\begin{aligned} \langle \phi^i(x) - \phi^i(y), \phi^j(x) - \phi^j(y) \rangle_t &= \langle \phi^i(x), \phi^j(x) \rangle_t + \langle \phi^i(y), \phi^j(y) \rangle_t - 2\langle \phi^i(x), \phi^j(y) \rangle_t \\ &= 2 \int_0^t (\delta_{ij} - b_{i,j}(\phi_s(x) - \phi_s(y))) ds. \end{aligned}$$

For arbitrary $f \in C_0^2(\mathbb{R}^d : \mathbb{R})$ with the help of Itô's formula we obtain

$$\begin{aligned}
f(\phi_t(x) - \phi_t(y)) - f(x - y) &= \sum_{i=1}^d \int_0^t \partial_i f(\phi_s(x) - \phi_s(y)) d(\phi_s^i(x) - \phi_s^i(y)) \\
&+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij}^2 f(\phi_s(x) - \phi_s(y)) d\langle \phi_s^i(x) - \phi_s^i(y), \phi_s^j(x) - \phi_s^j(y) \rangle_s \\
&= \sum_{i=1}^d \int_0^t \partial_i f(\phi_s(x) - \phi_s(y)) F^i(ds, \phi_s(x)) - \sum_{i=1}^d \int_0^t \partial_i f(\phi_s(x) - \phi_s(y)) F^i(ds, \phi_s(y)) \\
&- c \sum_{i=1}^d (\phi_t^i(x) - \phi_t^i(y)) \partial_i f(\phi_s(x) - \phi_s(y)) ds \\
&+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t 2(\delta_{ij} - b_{i,j}(\phi_s(x) - \phi_s(y))) \partial_{ij}^2 f(\phi_s(x) - \phi_s(y)) ds,
\end{aligned}$$

that is $t \mapsto f(\phi_t(x) - \phi_t(y)) - f(x - y) - \int_0^t (\mathcal{L}_d f)(\phi_s(x) - \phi_s(y)) ds$ is a continuous martingale

for all $x \in \mathbb{R}^d$ and we can conclude as before.

(d) Let $x \neq y \in \mathbb{R}^d$ be arbitrary but fixed.

Denote by r_t the distance process $|\phi_t(x) - \phi_t(y)|$. The homeomorphic property of ϕ implies that $\mathbb{P}(r_t > 0, \text{ for all } t \in \mathbb{R}_+) = 1$. Set $q_t := \phi_t(x) - \phi_t(y)$. For arbitrary $f \in C_0^2(\mathbb{R} : \mathbb{R})$ the Itô formula implies that

$$\begin{aligned}
f(r_t) - f(r_0) &= \underbrace{\sum_{i=1}^d \int_0^t f'(r_s) \frac{q_s^i}{r_s} F^i(ds, \phi_s(x)) - \sum_{i=1}^d \int_0^t f'(r_s) \frac{q_s^i}{r_s} F^i(ds, \phi_s(y))}_{M_t} \\
&+ \int_0^t \left[(-cr_s + \frac{d-1}{r_s} - \sum_{i=1}^d \frac{b_{i,i}(q_s)}{r_s} + \sum_{i,j=1}^d \frac{q_s^i q_s^j}{r_s^3} b_{i,j}(q_s)) f'(r_s) + (1 - \sum_{i,j=1}^d b_{i,j}(q_s) \frac{q_s^i q_s^j}{r_s^2}) f''(r_s) \right] ds.
\end{aligned}$$

Using the equalities

$$\begin{aligned}
\sum_{i,j=1}^d b_{i,j}(x) x^i x^j &= (B_L(|x|) - B_N(|x|)) \sum_{i,j=1}^d \frac{(x^i x^j)^2}{|x|^2} + \sum_{i,j=1}^d \delta_{ij} B_N(|x|) x^i x^j \\
&= (B_L(|x|) - B_N(|x|)) |x|^2 + B_N(|x|) |x|^2 = B_L(|x|) |x|^2
\end{aligned}$$

and

$$\sum_{i=1}^d b_{i,i}(x) = (B_L(|x|) - B_N(|x|)) \sum_{i=1}^d \frac{(x^i)^2}{|x|^2} + \sum_{i=1}^d \delta_{ii} B_N(|x|) = B_L(|x|) + (d-1)B_N(|x|)$$

we obtain

$$f(r_t) - f(r_0) = M_t + \int_0^t \left[((d-1) \frac{1 - B_N(r_s)}{r_s} - c r_s) f'(r_s) + (1 - B_L(r_s)) f''(r_s) \right] ds,$$

that is

$$f(r_t) - f(r_0) - \int_0^t (\mathcal{A}f)(r_s) ds$$

is a continuous martingale and standard results about the martingale problem imply that $(r_t)_{t \geq 0}$ is a diffusion with generator \mathcal{A} . \square

The following corollary will be used throughout the paper. It has been proven in a very general setting by Cranston, Scheutzow and Steinsaltz in [5], however they impose too strong boundedness assumptions on the local characteristic of the generating field V to be directly applicable to the case of an IOUF. In particular, they assume that the drift is uniformly bounded which is obviously not fulfilled in our case. Their proof however does work under much milder conditions which include the case of an IOUF. Therefore, their statement formally does not cover our case, however their proof does. Recent joint work of one of the authors and Michael Scheutzow contains the following bound under milder conditions making it directly applicable to our case. For the sake of completeness we give the proof for the case of an IOUF.

Corollary 2.3. *Let ϕ be an IOUF and let $x, y \in \mathbb{R}^d$, $x \neq y$ be arbitrary. Then there is a standard Brownian motion $\{B_t : t \geq 0\}$ (depending on x and y !) and constants $\sigma, \lambda > 0$ such that \mathbb{P} -almost surely*

$$|\phi_t(x) - \phi_t(y)| \leq |x - y| \exp \{ \sigma B_t^* + \lambda t \} \quad \text{for all } t \geq 0,$$

where as usual $B_t^* := \sup_{s \leq t} B_s$.

Proof. According to Proposition 2.2 (d) we can view $\{r_t := |\phi_t(x) - \phi_t(y)| : t \geq 0\}$ as the unique strong solution of

$$r_t = |x - y| + \int_0^t \sqrt{2(1 - B_L(r_s))} dW_s + \int_0^t \left((d-1) \frac{1 - B_N(r_s)}{r_s} - c r_s \right) ds.$$

where $\{W_t : t \geq 0\}$ is a standard Brownian motion (depending on x and y !). The asymptotic expansions of B_L and B_N around zero (see [4]) provide constants $a, b > 0$, such that

$$\sup_{u \geq 0} \frac{1 - B_N(u)}{u^2} \leq a \quad \text{and} \quad \sup_{u \geq 0} \frac{1 - B_L(u)}{u^2} \leq b. \quad (2.3)$$

With

$$M_t = \int_0^t \sqrt{\frac{2(1 - B_L(r_s))}{r_s^2}} dW_s$$

the SDE for r_t can be rewritten as

$$r_t = |x - y| + \int_0^t r_s dM_s + \int_0^t \left((d-1) \frac{1 - B_N(r_s)}{r_s^2} - c \right) r_s ds.$$

Consider the process (Y_t) , solving the SDE

$$Y_t = |x - y| + \int_0^t Y_s dM_s + \int_0^t ((d-1)a - c) Y_s ds,$$

that is

$$Y_t = |x - y| \exp \left\{ M_t + t((d-1)a - c) - \frac{1}{2} \langle M \rangle_t \right\}.$$

An application of the comparison theorem as stated in [25] (Theorem 54 on page 324, Chapter V, Section 9) implies

$$\mathbb{P}(r_t \leq Y_t \text{ for all } t \geq 0) = 1, \quad (2.4)$$

as we have $(d-1) \frac{1 - B_N(u)}{u^2} - c \leq (d-1)a - c$. According to (2.3) for the quadratic variation of M we have $\langle M \rangle_t \leq 2bt$. Recall that there is a standard Brownian motion $(B_t)_{t \geq 0}$ such that $\mathbb{P}(M_t = B_{\langle M \rangle_t} \text{ for all } t < \infty) = 1$. Thus for Y_t we get \mathbb{P} -almost surely:

$$\begin{aligned} Y_t &\leq |x - y| \exp [B_{\langle M \rangle_t} + t((d-1)a - c)] \leq |x - y| \exp [B_{\langle M \rangle_t}^* + t((d-1)a - c)] \\ &\leq |x - y| \exp [B_{2bt}^* + t((d-1)a - c)] = |x - y| \exp \left[\sqrt{2b} \sup_{s \leq t} \tilde{B}_s + t((d-1)a - c) \right], \end{aligned}$$

for the Brownian motion $\tilde{B}_s = (2s)^{-\frac{1}{2}} B_{2bs}$, In view of (2.4) we are done \square

In the next proposition we give the Lyapunov spectrum of an isotropic Ornstein-Uhlenbeck flow. Actually strictly speaking we are allowed to speak about Lyapunov exponents only for random dynamical systems (RDS). However to a stochastic flow ϕ with stationary increments generated via a Kunita type SDE and satisfying some regularity conditions one can canonically construct a corresponding two-sided RDS φ having the same distribution. The proof can be found in [2] without the extension from one sided time to \mathbb{R} , which is however canonical. We make this precise for an IOUF in the following remark.

Remark 2.4. Let ϕ be an IOUF. There is a perfect cocycle φ on \mathbb{R}^d with time \mathbb{R} over some metric dynamical system $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\theta_t)_{t \in \mathbb{R}}, \tilde{\mathbb{P}})$ such that the distributions of $\{\phi_{s,t} : s, t \geq 0\}$ and $\{\varphi(t-s, \theta_s \tilde{\omega}) : s, t \geq 0\}$ coincide.

From now on we shall identify the flow with the associated RDS.

Proposition 2.5. *The isotropic Ornstein-Uhlenbeck flow ϕ in \mathbb{R}^d has d different Lyapunov exponents, which are given by*

$$\lambda_i := (d - i) \frac{\beta_N}{2} - i \frac{\beta_L}{2} - c.$$

In particular they all have simple multiplicities.

Proof. Consider the isotropic Brownian flow ψ , generated by the field F via the SDE

$$\psi_{s,t}(x) = x + \int_s^t F(du, \psi_{s,u}(x)) \text{ for all } 0 \leq s \leq t, x \in \mathbb{R}^d$$

and the IOUF ϕ :

$$\phi_{s,t}(x) = x + \int_s^t F(du, \phi_{s,u}(x)) - c \int_s^t \phi_{s,u}(x) du.$$

Let $D_x F(t, x)$ denote the Jacobi matrix of $F(t, x)$. Recall the convention $\phi_t := \phi_{0,t}$ and $\psi_t := \psi_{0,t}$. According to Theorem 3.3.4 from [17] we have

$$D_x \psi_t(x) = \text{Id}_{\mathbb{R}^d} + \int_0^t D_x F(du, \psi_u(x)) D_x \psi_u(x)$$

and

$$D_x \phi_t(x) = \text{Id}_{\mathbb{R}^d} + \int_0^t D_x F(du, \phi_u(x)) D_x \phi_u(x) - c \int_0^t D_x \phi_u(x) du.$$

Set $\gamma_t(x) := e^{-ct} D_x \psi_t(x)$. Itô's formula implies that γ solves the SDE

$$\gamma_t(x) = \text{Id}_{\mathbb{R}^d} + \int_0^t D_x F(ds, \psi_s(x)) \gamma_s(x) - c \int_0^t \gamma_s(x) ds.$$

For arbitrary but fixed $x \in \mathbb{R}^d$ consider the martingale fields

$$M^x(t, y) = \int_0^t D_x F(ds, \phi_s(x)) y : \mathbb{R}_+ \times \mathbb{R}^{d \times d} \times \Omega \rightarrow \mathbb{R}^{d \times d} \text{ and}$$

$$N^x(t, y) = \int_0^t D_x F(ds, \psi_s(x)) y : \mathbb{R}_+ \times \mathbb{R}^{d \times d} \times \Omega \rightarrow \mathbb{R}^{d \times d}.$$

Straightforward calculation of their quadratic variations shows that

$$\langle M_{ij}^x(\cdot, y), M_{pq}^x(\cdot, z) \rangle_t = \langle N_{ij}^x(\cdot, y), N_{pq}^x(\cdot, z) \rangle_t = t \sum_{k,l=1}^d \partial_k \partial_l b_{i,p}(0) y_{kj} y_{lq}$$

and therefore N^x and M^x have the same Gaussian distribution. Clearly, the equations for $e^{-ct} D_x \psi_t(x)$ and $D_x \phi_t(x)$ can be written with the help of the fields M^x and N^x :

$$\begin{aligned} \gamma_t(x) &= \text{Id}_{\mathbb{R}^d} + \int_0^t M^x(ds, \gamma_s(x)) - c \int_0^t \gamma_s(x) ds, \\ D_x \phi_t(x) &= \text{Id}_{\mathbb{R}^d} + \int_0^t N^x(ds, D_x \phi_s(x)) - c \int_0^t D_x \phi_s(x) ds \end{aligned} \tag{2.5}$$

and thus $\{\gamma_t(x) : t \geq 0\}$ and $\{D_x \phi_t(x) : t \geq 0\}$ have the same distributions for all $x \in \mathbb{R}^d$. The Lyapunov exponents of ψ have simple multiplicities and are given by $(d-i)\frac{\beta_N}{2} - i\frac{\beta_L}{2}$ (see [4]) and therefore almost surely

$$\lim_{t \rightarrow +\infty} \frac{1}{2t} \log [\sigma_i(D_x \psi_t(x))] = (d-i)\frac{\beta_N}{2} - i\frac{\beta_L}{2},$$

where $\sigma_i(M)$ denotes the i -th characteristic value of the matrix M (the i -th eigenvalue of $M^T M$). According to the multiplicative ergodic theorem the corresponding a.s. limits for ϕ also exist (see [23]) and satisfy

$$\begin{aligned} \lambda_i &:= \lim_{t \rightarrow +\infty} \frac{1}{2t} \log(\sigma_i(D_x \phi_t(x))) = \lim_{t \rightarrow +\infty} \frac{1}{2t} \log(\sigma_i(\gamma_t(x))) = \lim_{t \rightarrow +\infty} \frac{1}{2t} \log(\sigma_i(D_x \psi_t(x))) - c \\ &= (d-i)\frac{\beta_N}{2} - i\frac{\beta_L}{2} - c, \end{aligned}$$

□

In the last proposition in this section we give the transience/recurrence modes of the \mathbb{R}_+ -valued diffusion $r_t(x, y) := |\phi_t(x) - \phi_t(y)|$. The strong inwards drift clearly implies $\mathbb{P}(\lim_{t \rightarrow \infty} |\phi_t(x) - \phi_t(y)| = +\infty) = 0$ and therefore “transience” means almost sure convergence to zero, i.e. $\mathbb{P}(\lim_{t \rightarrow \infty} r_t(x, y) = 0) = 1$.

Proposition 2.6. *Let λ_1 denote the top Lyapunov exponent of the IOUF ϕ . Then for arbitrary $x, y \in \mathbb{R}^d$ the diffusion $r_t = r_t(x, y)$ on $(0, +\infty)$ is*

(a) *recurrent if $\lambda_1 \geq 0$, i.e.*

$$\mathbb{P}(\liminf_{t \rightarrow \infty} r_t(x, y) = 0) = \mathbb{P}(\limsup_{t \rightarrow \infty} r_t(x, y) = +\infty) = 1$$

(b) *transient if $\lambda_1 < 0$, that is*

$$\mathbb{P}(\lim_{t \rightarrow \infty} r_t(x, y) = 0) = 1.$$

Further, the speed measure of the diffusion r_t is finite if and only if $\lambda > 0$. In this case the unique invariant probability measure of r_t is given by

$$\frac{1}{\int_0^\infty m(u)du} m(x) dx, \quad \text{where} \quad m(x) = \frac{1}{1 - B_L(x)} \exp \left[\int_1^x \frac{(d-1)(1 - B_N(y)) - cy^2}{y(1 - B_L(y))} dy \right].$$

Proof. Observe that the exit time $S := \inf\{t \geq 0 : r_t(x, y) = 0 \text{ or } r_t(x, y) = +\infty\}$ is infinite almost surely. Indeed, for $x \neq y$, $\mathbb{P}(r_t(x, y) > 0 \text{ for all } t \geq 0) = 1$ by the homeomorphic property of the flow ϕ . Clearly $\mathbb{P}(r_t(x, y) < +\infty \text{ for all } t \geq 0) = 1$. The density $m(x)$ of the *speed measure* (see [14], Section 5.5.C, page 343) of $r_t(x, y)$ with respect to the Lebesgue measure satisfies the equation $\mathcal{A}^*m = 0$ and can be calculated explicitly (see e.g. [14], Section 5.5.C, eq. 5.51). Up to a multiplicative constant it is given by

$$m(x) = \frac{1}{1 - B_L(x)} \exp \left[\int_1^x \frac{(d-1)(1 - B_N(y)) - cy^2}{y(1 - B_L(y))} dy \right].$$

The *scale function* (see [14], Section 5.5.C, page 343) $s(x)$ is a solution of $\mathcal{A}u = 0$ and up to multiplicative and additive constants is given by (see e.g. [14], Section 5.5.B, eq. 5.42)

$$s(x) = \int_1^x \exp \left[- \int_1^y \frac{(d-1)(1 - B_N(z)) - cz^2}{z(1 - B_L(z))} dz \right] dy.$$

Observe that since there is a $\delta \in (0, 1)$, such that $-1 + \delta < B_L(z), B_N(z) < 1 - \delta$ for all $z \geq 1$ (see [4], Remark 2.18) we have for all $z \geq 1$

$$- \frac{(d-1)(1 - B_N(z)) - cz^2}{z(1 - B_L(z))} = - \frac{(d-1)(1 - B_N(z))}{z(1 - B_L(z))} + \frac{cz}{1 - B_L(z)} \geq - \frac{2(d-1)}{\delta z} + \frac{c}{2}z. \quad (2.6)$$

Therefore for $x \geq 1$

$$s(x) \geq \int_1^x \exp \left[\int_1^y \frac{c}{2}z - \frac{2(d-1)}{\delta z} dz \right] dy = \int_1^x \exp \left[\frac{c}{4}y^2 - \frac{2(d-1)}{\delta} \ln(y) - \frac{c}{4} \right] dy \xrightarrow{x \rightarrow +\infty} +\infty. \quad (2.7)$$

In order to obtain the asymptotics of $s(x)$ as $x \rightarrow +0$, recall that for $r \rightarrow 0$ we have (see [4])

$$B_L(r) = 1 - \frac{\beta_L}{2}r^2 + O(r^4) \quad \text{and} \quad B_N(r) = 1 - \frac{\beta_N}{2}r^2 + O(r^4)$$

and therefore

$$- \frac{(d-1)(1 - B_N(z)) - cz^2}{z(1 - B_L(z))} = - \frac{1}{z} \frac{(d-1)\frac{\beta_N}{2} - c + O(z^2)}{\frac{\beta_L}{2} + O(z^2)}.$$

Since

$$\frac{(d-1)\frac{\beta_N}{2} - c + O(z^2)}{\frac{\beta_L}{2} + O(z^2)} = \frac{(d-1)\frac{\beta_N}{2} - c}{\frac{\beta_L}{2}} + O(z^2)$$

we have that for $x \rightarrow 0$,

$$s'(x) \sim \exp \left[- \frac{(d-1)\frac{\beta_N}{2} - c}{\frac{\beta_L}{2}} \int_1^x \frac{1}{z} dz \right] = x^{-\frac{(d-1)\frac{\beta_N}{2} - c}{\frac{\beta_L}{2}}},$$

which is integrable around zero if and only if

$$\frac{(d-1)\frac{\beta_N}{2} - c}{\frac{\beta_L}{2}} < 1 \iff \lambda_1 < 0. \quad (2.8)$$

We now consider the density of the speed measure $m(x)$. Similarly as above, for all $x \geq 1$ we have $-1 + \delta < B_L(x), B_N(x) < 1 - \delta$ and therefore

$$\begin{aligned} m(x) &= \frac{1}{1 - B_L(x)} \exp \left[\int_1^x \frac{(d-1)(1 - B_N(y)) - cy^2}{y(1 - B_L(y))} dy \right] \leq \frac{1}{\delta} \exp \left[\int_1^x \frac{2(d-1)}{\delta} \frac{1}{y} - \frac{c}{2} y dy \right] \\ &= \frac{1}{\delta} \exp \left[\frac{2(d-1)}{\delta} \ln(x) - \frac{c}{4} x^2 + \frac{c}{4} \right], \end{aligned}$$

which is certainly integrable around $+\infty$.

For $x \rightarrow +0$

$$m(x) \sim \frac{1}{\frac{\beta_L}{2} x^2} \exp \left[\frac{(d-1)\frac{\beta_N}{2} - c}{\frac{\beta_L}{2}} \int_1^x \frac{1}{y} dy \right] = \frac{2}{\beta_L} x^{\frac{(d-1)\frac{\beta_N}{2} - c}{\frac{\beta_L}{2}} - 2},$$

which is integrable around zero if and only if

$$\frac{(d-1)\frac{\beta_N}{2} - c}{\frac{\beta_L}{2}} - 2 > -1 \iff \frac{(d-1)\frac{\beta_N}{2} - c}{\frac{\beta_L}{2}} > 1 \Leftrightarrow \lambda_1 > 0.$$

In the case $\lambda_1 = 0$ the speed measure puts infinite charge on any interval of the form $[0, \epsilon]$. Combining these results about the behavior of the scale function near zero and infinity we get (see [14], Proposition 5.5.22)

1. (Transience) If $\lambda_1 < 0$ then $s(+\infty) = +\infty$ and $s(+0) > -\infty$, and therefore

$$\mathbb{P} \left(\lim_{t \rightarrow +\infty} r_t = 0 \right) = \mathbb{P} \left(\sup_{t \in \mathbb{R}_+} r_t < +\infty \right) = 1.$$

2. (Recurrence) If $\lambda_1 \geq 0$ then $s(+\infty) = +\infty$, $s(+0) = -\infty$. In this case we have

$$\mathbb{P} \left(\inf_{t \in \mathbb{R}_+} r_t = 0 \right) = \mathbb{P} \left(\sup_{t \in \mathbb{R}_+} r_t = +\infty \right) = 1.$$

Since the speed measure is integrable around zero exactly when $\lambda_1 > 0$ and is always integrable around $+\infty$ we have that

$$\int_0^{+\infty} m(x) dx < \infty \iff \lambda_1 > 0$$

and therefore the measure

$$m_p(x)dx := \frac{1}{\int_0^{+\infty} m(u) du} m(x) dx$$

is a well defined probability measure which is invariant as $\mathcal{A}^* m_p(x) = 0$. In case $\lambda_1 \leq 0$ the speed measure is not finite and we do not have any invariant probability measures. \square

We finish this section by proving the following lemma concerning the a.s. spatial regularity of IOUFs. The one-point-motion $\phi_t(x)$ of an IOUF is as already stated an Ornstein-Uhlenbeck-process and so if x is far away from the origin $|\phi_t(x)|$ will decrease roughly as $|x|e^{-ct}$ (which is the expected decrease). because the variance of $|\phi_t(x)|$ is rather negligible for small t . We may expect the IOUFs unitstep-discretisation $\phi = \phi_{0,1}$ to look like e^{-c} time the Identity on a large scale. For a fixed x this is quite obvious but care has to be taken about the fact that we are dealing with infinitely many random variables. The next lemma states that this is no problem at all.

Lemma 2.7. *Let $\phi = \phi_{0,1} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be as in Definition 1.1 (with $s = 0$ and $t = 1$). Then we have a.s.*

1.

$$\lim_{R \rightarrow \infty} \sup_{\|x\| \geq R} \frac{\|\phi(x) - e^{-c}x\|}{\|x\|} = 0 \quad (2.9)$$

2.

$$\lim_{R \rightarrow \infty} \sup_{\|x\| \geq R} \frac{\|\phi(x)\|}{\|x\|} = e^{-c} \quad (2.10)$$

Proof. Observe that the second formula is an easy consequence of the first one, so we will only have to prove (2.9). Therefore it is sufficient to show

$$\lim_{R \rightarrow \infty} \sup_{R \leq \|x\| \leq R+1} \frac{\|\phi(x) - e^{-c}x\|}{\|x\|} = 0 \quad (2.11)$$

Let $A_R := \{x \in \mathbb{R}^d : R \leq \|x\| \leq R+1\}$. We want to apply the chaining technique (see Proposition 4.9) and so we first observe that A_R can be covered by $c_3 R^d 3^j$ balls of radius $\delta_j = 3^{-j}$. Let χ_j consist of the centers of these balls and $\epsilon_j = 2^{-j-2}$ as well as $\epsilon = \frac{1}{2}$. Fixing $\tilde{\epsilon} > 0$ and an arbitrary $x_0 \in A_R$ and assuming $R \geq \frac{16e^c}{\tilde{\epsilon}}$ we conclude

$$\begin{aligned} \mathbb{P} \left[\sup_{x \in A_R} \frac{\|\phi(x) - e^{-c}x\|}{\|x\|} > \tilde{\epsilon} \right] &\leq \mathbb{P} \left[\sup_{x \in A_R} \|\phi(x) - e^{-c}x\| > \tilde{\epsilon}R \right] \leq \mathbb{P} \left[\|\phi(x_0) - e^{-c}x_0\| > \frac{\tilde{\epsilon}R}{2} \right] \\ &+ \sum_{j=0}^{\infty} c_3 R^d 3^j \sup_{|x-y| \leq 3^{-j}} \mathbb{P} [\|\phi(x) - e^{-c}x\| - \|\phi(y) - e^{-c}y\| > 2^{-j-2}\tilde{\epsilon}R] \end{aligned}$$

Using the standard estimates for the normal distribution (as stated in Lemma 4.10 for a Brownian motion) one easily shows that.

$$\mathbb{P} \left[\|\phi(x_0) - e^{-c}x_0\| > \frac{\tilde{\epsilon}R}{2} \right] \leq c_4 e^{-\frac{\tilde{\epsilon}^2}{8d^2} R^2}$$

where c_4 is a constant only depending on c and d . Using Corollary 2.3 and Lemma 4.10 we get for $|x - y| \leq 3^{-j}$:

$$\begin{aligned}
& \mathbb{P} [||\phi(x) - e^{-c}x|| - ||\phi(y) - e^{-c}y|| > 2^{-j-2}\tilde{\epsilon}R] \\
& \leq \mathbb{P} [||\phi(x) - e^{-c}x|| - ||\phi(y) - e^{-c}y|| > 2^{-j-2}\tilde{\epsilon}R3^j|x - y|] \\
& \leq \mathbb{P} [||\phi(x) - \phi(y)|| > 2^{-j-3}\tilde{\epsilon}R3^j|x - y|] \\
& \leq \mathbb{P} \left[B_1^* \geq \frac{\log(2^{-3-j}\tilde{\epsilon}R3^j) - \lambda}{\sigma} \right] \leq c_5(2^{-j-3}\tilde{\epsilon}R3^j)^{-\frac{\log(2^{-3-j}\tilde{\epsilon}R3^j) - 2\lambda}{2\sigma^2}}
\end{aligned}$$

where the constant c_5 depends only on σ and λ . Combining all the above estimates we conclude that for arbitrary $\tilde{\epsilon} > 0$ we have that $\mathbb{P} \left[\sup_{x \in \mathcal{A}_R} \frac{||\phi(x) - e^{-c}x||}{||x||} > \tilde{\epsilon} \right]$ is sumable over R which yields the desired conclusion via an application of the Borel-Cantelli Lemma. \square

3 The dimension of the statistical equilibrium of an isotropic Ornstein-Uhlenbeck flow

Throughout this section $\varphi: \mathbb{R} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ will denote an RDS over the MDS $(\Omega, \mathcal{F}, (\theta_t)_{t \in \mathbb{R}}, \mathbb{P})$, associated, in the sense of Remark 2.4, to an isotropic Ornstein-Uhlenbeck stochastic flow $\phi: \mathbb{R}_+ \times \mathbb{R}_+ \times \tilde{\Omega} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$. We will also call φ an IOUF. The Lyapunov spectrum of φ will be denoted by $\lambda_1 > \dots > \lambda_d$. It has been calculated in Proposition 2.5.

We also have the standing assumption that the top Lyapunov exponent is strictly positive.

Our goal here will be to calculate the dimension of the statistical equilibrium of φ .

In the next theorem we quote some general facts concerning the statistical equilibrium of an IOUF. In a slightly different setting it can be found in [18], [3] and [17], more precisely in [18] for the case φ is an IBF, in [3] in the case φ is generated by an SDE driven by finitely many Brownian motions. Virtually the same is also the proof of Theorem 4.3.6 in [17], however the measures considered there are given by $m_t(A) := m(\varphi(t, A, \omega))$ for positive t .

Theorem 3.1. *Let $\varphi: \mathbb{R} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ be an RDS over the MDS $(\Omega, \mathcal{F}, (\theta_t)_{t \in \mathbb{R}}, \mathbb{P})$, associated to an isotropic Ornstein-Uhlenbeck stochastic flow ϕ in the sense of Remark 2.4. Then the following statements hold:*

- (a) *There is an invariant Markovian measure (see Proposition 1.4.3 in [1]) μ on $(\Omega \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d))$ with factorization*

$$\mu(dx, d\omega) = \mu_\omega(dx) \mathbb{P}(d\omega)$$

satisfying

$$\mathbb{E} \mathbb{P} \mu_\omega = \nu, \quad \text{i.e.} \quad \mathbb{E} \int f d\mu_\omega = \int f d\nu \text{ for all } f \in C_b(\mathbb{R}^d),$$

where ν is the unique invariant probability measure for the one-point motion and $C_b(\mathbb{R}^d)$ is the set of all bounded continuous functions mapping from \mathbb{R}^d to \mathbb{R} .

- (b) *The measure μ_ω can be obtained as the \mathbb{P} -almost sure weak limit*

$$\lim_{t \rightarrow \infty} \nu \circ \varphi^{-1}(t, \theta_{-t}\omega) = \mu_\omega.$$

(c) For arbitrary $t \in \mathbb{R}$ it holds that

$$\mu_\omega \circ \varphi^{-1}(t, \omega) = \mu_{\theta_t \omega} \quad \mathbb{P}\text{-almost surely.}$$

Remark 3.2. It is well known that

$$\nu(dx) = \left(\frac{c}{\pi}\right)^{\frac{d}{2}} e^{-c|x|^2}.$$

Before we proceed with proving the main result in this section we need to introduce the notion of a *pointwise dimension* of a measure, which is in a sense a refinement of the notion of density.

Definition 3.3. Let μ be a Borel measure on \mathbb{R}^d . The *lower and upper pointwise dimensions* are defined by

$$\underline{d}_\mu(x) = \liminf_{\epsilon \rightarrow +0} \frac{\log \mu(B(x, \epsilon))}{\log \epsilon} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{\epsilon \rightarrow +0} \frac{\log \mu(B(x, \epsilon))}{\log \epsilon}.$$

If $\underline{d}_\mu(x) = \overline{d}_\mu(x)$ we call the common value the *pointwise dimension* and denote it by $d_\mu(x)$. If there is a constant $d(\mu) = d_\mu(x)$ for μ -almost all x then $d(\mu)$ is called the dimension of the measure μ .

One can easily construct quite regular measures which do not have fixed dimension. For example in \mathbb{R}^2 take a disjoint ball B and a line segment L and place uniform distributions U_B and U_L on them. U_B and U_L have dimensions 2 and 1 respectively, but the dimension of $U := U_B + U_L$ is not well defined. Measures having a fixed dimension are quite special and in a sense look similar around almost every point in a set supporting the measure. The main result in this chapter is that the statistical equilibrium of an IOUF is such a special measure having constant pointwise dimension. Further we give an explicit expression for the dimension using the results of F. Ledrappier and L.-S. Young in [22]. There the authors calculate the dimension of the statistical equilibrium of a composition of independent and identically distributed diffeomorphisms on a compact space. There is no obstacle in the fact that we have an RDS in continuous time since the mappings $(\varphi(1, \theta_z \omega))_{z \in \mathbb{Z}}$ are independent and identically distributed and define the same statistical equilibrium.

The only fact we still need to prove is that their assumption that the state space is compact is also not an obstacle since we have an invariant probability measure for the one-point motion and the RDS has a weak attractor. This will be done in the course of the proof of the next theorem.

One can link the dimension of a probability measure μ on $\mathcal{B}(\mathbb{R}^d)$ (if it exists) to the minimal Hausdorff dimension of a set supporting μ :

$$d(\mu) = \inf \{ \dim_H Y : Y \in \mathcal{B}(\mathbb{R}^d), \mu(Y) = 1 \}, \quad (3.12)$$

where $\dim_H(Y)$ denotes the Hausdorff dimension of the set Y . For the proof of this fact see e.g. [27]. Observe that the above equality does not concern *the support* of μ since we are not allowed to take the closure of the set Y . For example consider a probability measure μ on \mathbb{R} with $\mu([0, 1] \cap \mathbb{Q}) = 1$ supported on $[0, 1]$. The Hausdorff dimension of the support is 1, while in fact μ is carried by a countable set and thus has dimension zero.

We follow [22] and introduce the *Lyapunov dimension*:

Definition 3.4. Let $\lambda_1 > \dots > \lambda_r$ be the Lyapunov exponents and m_1, \dots, m_r the corresponding multiplicities of some RDS on a d -dimensional smooth (C^∞) manifold. Let k be the largest integer, such that $\sum_{i=1}^k \lambda_i m_i > 0$. The corresponding *Lyapunov dimension* $\mathcal{D}(\lambda_1, \dots, \lambda_r)$ is defined as follows:

$$\mathcal{D}(\lambda_1, \dots, \lambda_r) = \begin{cases} d & \text{if } \sum_{i=1}^k m_i = d \\ \sum_{i=1}^k m_i - \frac{1}{\lambda_{k+1}} \sum_{i=1}^k \lambda_i m_i & \text{otherwise.} \end{cases}$$

Theorem 3.5. Let $\varphi : \mathbb{R} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ be an RDS over the MDS $(\Omega, \mathcal{F}, (\theta_t)_{t \in \mathbb{R}}, \mathbb{P})$ associated to an isotropic Ornstein-Uhlenbeck stochastic flow with strictly positive top Lyapunov exponent. The dimension of μ_ω exists for almost all ω , is deterministic and one has

$$d(\mu_\omega) = \mathcal{D}(\lambda_1, \dots, \lambda_d).$$

Proof:

Let us first summarize the simple idea: Consider a one-point compactification of \mathbb{R}^d . We will associate a cocycle ψ on the compactified space, which has the same Lyapunov spectrum as φ . Further, the dimension of the nontrivial statistical equilibrium of the new RDS will coincide with the one of φ . The dimension of the statistical equilibrium of ψ has been calculated in [22].

Let \mathcal{S} be a d -dimensional sphere in \mathbb{R}^{d+1} equipped with the induced topology from \mathbb{R}^{d+1} , and let N be its north pole. Let $\mathcal{S}^N := \mathcal{S} \setminus \{N\}$.

It is well known that there is a smooth (C^∞) diffeomorphic map

$$g : \mathbb{R}^d \cup \{\infty\} \rightarrow \mathcal{S} \text{ such that } g(\mathbb{R}^d) = \mathcal{S} \setminus \{N\}.$$

As usual we will denote by $T_p M$ the tangent space of the manifold M at the point $p \in M$, by TM the tangent bundle of the manifold M and by Df the differential of a differentiable mapping $f : M \rightarrow N$, i.e $Df : TM \rightarrow TN$ is given by

$$Df(x, \cdot) : T_x M \rightarrow T_{f(x)} N, \quad (Df)(x, v) = (f(x), Df(x)v).$$

Consider the RDS ψ^N on \mathcal{S}^N over the metric DS $(\theta)_t$ given by

$$\psi^N(t, y, \omega) : \mathbb{R} \times \mathcal{S}^N \times \Omega \rightarrow \mathcal{S}^N \quad y \rightarrow g \circ \varphi(t, \omega) \circ g^{-1}.$$

That is, ψ^N is cohomologous to φ with cohomology g (see Proposition 1.9.6 on page 46 in [1]). We will show that ψ^N has the same Lyapunov spectrum as φ by showing that their linearizations are Lyapunov cohomologous with cohomology Dg (see Proposition 4.1.9 in [1]). According to Proposition 4.2.5 in [1], the linearization T_1 of ψ^N is a bundle RDS on $\Omega \times T\mathcal{S}^N$

$$T_1 : \Omega \times T\mathcal{S}^N \rightarrow \Omega \times T\mathcal{S}^N \quad (\omega, (y, v)) \rightarrow (\theta_t \omega, D\psi^N(t, \omega)(y, v))$$

over the skew-product shift

$$\Theta_t : \Omega \times \mathcal{S}^N \rightarrow \Omega \times \mathcal{S}^N, \quad \Theta_t(\omega, x) = (\theta_t \omega, \psi^N(t, \omega, x)).$$

T_1 is isomorphic (see Definition 1.9.8 in [1]) to the linearization T_2 of φ

$$T_2: \Omega \times T\mathbb{R}^d \rightarrow \Omega \times T\mathbb{R}^d \quad (\omega, (x, v)) \rightarrow (\theta_t \omega, D\varphi(t, \omega)(x, v))$$

via the mapping

$$K: \Omega \times T\mathbb{R}^d \rightarrow \Omega \times T\mathcal{S}^N \quad (\omega, (x, v)) \rightarrow (\omega, (g(x), Dg(x)v))$$

since obviously $K(\omega, x, \cdot): T_x\mathbb{R}^d \rightarrow T_{g(x)}\mathcal{S}^N$ is linear on fibers. T_2 is a bundle RDS over the skew-product shift

$$\tilde{\Theta}_t: \Omega \times \mathbb{R}^d \rightarrow \Omega \times \mathbb{R}^d, \quad \Theta_t(\omega, x) = (\theta_t \omega, \varphi(t, x, \omega)).$$

Since $g: \mathbb{R}^d \rightarrow \mathcal{S}^N$ is diffeomorphic, the functions

$$\begin{aligned} n_1: \Omega \times \mathbb{R}^d &\rightarrow (0, +\infty), \quad (\omega, x) \rightarrow \log \|K(\omega, x)\| = \log \|Dg(x)\| \\ n_2: \Omega \times \mathcal{S}^N &\rightarrow (0, +\infty), \quad (\omega, y) \rightarrow \log \|K^{-1}(\omega, y)\| = \log \|Dg^{-1}(y)\|, \end{aligned}$$

where $\|\cdot\|$ denotes the operator norm, are continuous. According to Proposition 4.1.9 in [1], in order to show that ψ^N has the same Lyapunov spectrum as φ we need to show that for $\mu := \mu_\omega(dx) \mathbb{P}(d\omega)$ -almost all (x, ω) and respectively $\mu_\omega \circ g^{-1}(dy) \mathbb{P}(d\omega)$ -almost all (y, ω)

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} \frac{1}{t} n_1(\tilde{\Theta}_t(\omega, x)) &= \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|Dg(\varphi(t, x, \omega))\| = 0 \quad \text{and} \\ \lim_{t \rightarrow \pm\infty} \frac{1}{t} n_2(\Theta_t(\omega, y)) &= \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|Dg^{-1}(\psi^N(t, y, \omega))\| = 0, \end{aligned} \tag{3.13}$$

which means that K is a *Lyapunov Cohomology* between the linearizations T_1 and T_2 (see Definition 4.1.6 in [1]). According to Proposition 4.1.3 on page 165 of [1] (Dichotomy for linear growth of stationary processes) it is enough to show that $\mu_\omega(dx) \mathbb{P}(d\omega)$ and respectively $\mu_\omega \circ g^{-1}(dy) \mathbb{P}(d\omega)$ -almost surely

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{1}{t} n_1(\tilde{\Theta}_t(\omega, x)) &= \lim_{t \rightarrow +\infty} \frac{1}{t} \log \|Dg(\varphi(t, x, \omega))\| = 0 \quad \text{and} \\ \lim_{t \rightarrow +\infty} \frac{1}{t} n_2(\Theta_t(\omega, y)) &= \lim_{t \rightarrow +\infty} \frac{1}{t} \log \|Dg^{-1}(\psi^N(t, y, \omega))\| = 0. \end{aligned} \tag{3.14}$$

However (3.14) holds true and the main reason is that the invariant measure μ has Markov nature, i.e. the factorization μ_ω is measurable with respect to the past ($\mathcal{F}_{-\infty}^0$ -measurable). First observe that for every fixed $x \in \mathbb{R}^d$ and $y \in \mathcal{S}^N$

$$\varphi(t, x) \xrightarrow[t \rightarrow \pm\infty]{} \mathcal{N}(0, \frac{1}{2c} \cdot \text{Id}) \text{ in distribution}$$

which follows from the convergence of an Ornstein-Uhlenbeck process to its equilibrium and

$$\psi^N(t, y) = g(\varphi(t, g^{-1}(y))) \xrightarrow[t \rightarrow \pm\infty]{} \mathcal{N}(0, \frac{1}{2c} \cdot \text{Id}) \circ g^{-1} \text{ in distribution}$$

because convergence in distribution is preserved by continuous mappings. Combining these with the continuity of $\log \|Dg(\cdot)\|$ and $\log \|Dg^{-1}(\cdot)\|$ implies that the convergences (3.14) hold for every fixed $x \in \mathbb{R}^d$ and $y \in \mathcal{S}^N$, \mathbb{P} -almost surely. Now, since μ_ω is independent of \mathcal{F}_0^t for every $t > 0$ we obtain that (3.14) holds also for $\mu_\omega(dx) \mathbb{P}(d\omega)$ -almost all (x, ω) and respectively for $\mu_\omega \circ g^{-1}(dy) \mathbb{P}(d\omega)$ -almost all (y, ω) . Therefore K is a Lyapunov cohomology and the Lyapunov spectrum of ψ^N is the same as the one of φ .

Now define the RDS ψ

$$\psi(t, y, \omega) = \begin{cases} \psi^N(t, y, \omega) & \text{if } y \neq N, \\ N & \text{if } y = N. \end{cases}$$

It is an RDS on \mathcal{S} and its linearization is a bundle RDS over the skew-product metric DS $\Theta_t(\omega, y) = (\theta_t \omega, \psi(t, y, \omega))$, preserving the measure $\mu_\omega \circ g^{-1}(dy) \mathbb{P}(d\omega)$. However, since this measure puts zero mass on the Θ_t -invariant set $\{N\} \times \Omega$, and ψ^N differs from ψ only on this set, we can conclude that the Lyapunov spectrum of the linearization of ψ^N coincides with the Lyapunov spectrum of the linearization of ψ , viewed as a bundle RDS over the metric DS Θ_t preserving $\mu_\omega \circ g^{-1}(dy) \mathbb{P}(d\omega)$.

Despite the fact that one can use the Lemma 2.7 to show the differentiability of ψ , it is easier to consider the two ergodic components of ψ , each of them being a C^2 random dynamical system. Once started in $\mathcal{S} \setminus N$ one stays there and hence the differentiability in N is not important for our purposes, since the flow couldn't even observe a lack of it.

Now we want to apply the results of Ledrappier and Young [22, Theorem A] giving the dimension of the statistical equilibrium of an ergodic random dynamical system and therefore to be in their setting we will consider the discretized cocycle $\psi: \Omega \times \mathbb{R}^d \times \mathbb{Z} \rightarrow \mathbb{R}^d$. In our case the assumptions in [22] are verified:

1. Clearly the shift $\theta := \theta_1$ is ergodic.
2. The set of invariant probability measures of the one-point motion of ψ is given by $\{\lambda \delta_N + (1 - \lambda) \mathcal{N}(0, \frac{1}{2c} \cdot \text{Id}) \circ g^{-1} : \lambda \in [0, 1]\}$ and thus $\nu \circ g^{-1} := \mathcal{N}(0, \frac{1}{2c} \cdot \text{Id}) \circ g^{-1}$ is an extremal point in this convex set.

Further, the technical assumptions

$$\mathbb{E} \log^+ \|\psi(1)\| < \infty \text{ and } \mathbb{E} \log^+ \|\psi(1)^{-1}\| < \infty$$

are direct consequences of Theorem 2.1(v) from [23], after taking into account that ψ acts on the compact space \mathcal{S} .

The last condition which has to be checked is that one of the Hypothesis A , A' or B from [22] is satisfied. It can be done for example using the remarks on stochastic flows in Section 5.2 of [22] by showing that the diffusion $(\varphi(t, x), D\varphi(t, x)e)$ is hypoelliptic and therefore the same holds for ψ .

We will check however the *Hypothesis A*:

3. The MET gives splitting of $T_y \mathcal{S}$ as direct sum $E_1(y, \omega) \oplus \dots \oplus E_m(y, \omega)$ of linear subspaces corresponding to the distinct Lyapunov exponents $\lambda_1 > \dots > \lambda_d$. The hypothesis states that for $\nu \circ g^{-1}$ -almost all y the distribution of $\omega \rightarrow E^j(y, \omega)$ for $j = k$ and $j = k + 1$, where k is as in the definition of the Lyapunov dimension, is absolutely continuous with respect to the Riemannian measure on the space of $\sum_{i \geq j} m_i$ -planes in $T_y \mathcal{S}$. Here m_i is the multiplicity of λ_i and $E^j(y, \omega) := \oplus_{i \geq j} E_i(y, \omega)$.

If we can verify the hypothesis for φ it will be also true for ψ^N on \mathcal{S}^N , because of Proposition

4.1.9 in [1] and the fact that φ and ψ^N are Lyapunov cohomologous. Further it will also hold for ψ as $\nu \circ g^{-1} \otimes \mathbb{P}(\delta_N \times \Omega) = 0$. However, for arbitrary orthogonal transformation $O \in \mathcal{O}_d$ the isotropy of φ implies that the distribution of $O\varphi(t, \cdot)$ coincides with the distribution of $\varphi(t, O \cdot)$ and therefore the distributions of $OD\varphi(t, 0)$ is the same as the one of $D\varphi(t, 0)$, and thus for the splitting $E_1(0, \omega) \oplus \dots \oplus E_d(0, \omega)$ of $T_0\mathbb{R}^d$ we have that the distribution of $\omega \rightarrow E^j(0, \omega)$ is exactly uniform on the space of $(d-j)$ -planes in $T_0\mathbb{R}^d$. Actually, this holds also for arbitrary $x \in \mathbb{R}^d$ since the distribution of the semimartingale field $M_x(t, y) := \int_0^t DF(s, \varphi(s, x))y - cy$ generating $D\varphi(t, x)$ via an SDE is independent of x and therefore coincides with the distribution of the semimartingale field generating $D\varphi(t, 0)$ (see (2.5) and the discussion before in the proof of Proposition 2.5).

The preceding argument depends heavily on the rotation invariance of the generating field. Much more general treatment on the laws of the Osceledets spaces of cocycles arising as solutions of linear SDE is provided by Peter Imkeller in [12] and [13].

Now we use [22, Theorem A] and state that the dimension of $\mu_\omega \circ g^{-1}$ exists \mathbb{P} -almost surely and

$$d(\mu_\omega \circ g^{-1}) = \mathcal{D}(\lambda_1, \dots, \lambda_d).$$

It remains to be shown that the dimension of μ_ω also exists and equals the dimension of $\mu_\omega \circ g^{-1}$. This is done via the following:

Claim: Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \rightarrow Y$ be a locally Lipschitz bijection, such that f^{-1} is also locally Lipschitz. Let further μ be a compactly supported positive measure on $(X, \mathcal{B}(X))$ with $d(\mu) = d$. Then the dimension of the image measure $\mu \circ f^{-1}$ exists and equals d .

Proof of the Claim:

Let $K > 0$ be such that

$$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) \text{ and } d_X(f^{-1}(y_1), f^{-1}(y_2)) \leq K d_Y(y_1, y_2)$$

for all $x_1, x_2 \in \text{supp}(\mu)$ and $y_1, y_2 \in \text{supp}(\mu \circ f^{-1})$. Let $B_Y(\tilde{y}, R)$ be a ball in Y with radius R and centered at $\tilde{y} \in Y$. Then obviously

$$f^{-1}(B_Y(\tilde{y}, R)) \subset B_X(f^{-1}(\tilde{y}), KR)$$

This implies

$$\begin{aligned} \limsup_{R \rightarrow 0} \frac{\log \mu \circ f^{-1}(B_Y(\tilde{y}, R))}{\log R} &= \limsup_{R \rightarrow 0} \frac{\log \mu(f^{-1}(B_Y(\tilde{y}, R)))}{\log R} \leq \limsup_{R \rightarrow 0} \frac{\log \mu(B_X(f^{-1}(\tilde{y}), KR))}{\log R} \\ &= \limsup_{R \rightarrow 0} \frac{\log \mu(B_X(f^{-1}(\tilde{y}), KR))}{\log(KR) - \log K} = d\mathbb{1}_{\text{supp}(\mu)}(f^{-1}(\tilde{y})) = d\mathbb{1}_{\text{supp}(\mu \circ f^{-1})}(\tilde{y}). \end{aligned}$$

On the other hand

$$B_X(f^{-1}(\tilde{y}), \frac{1}{K}R) \subset f^{-1}(B_Y(\tilde{y}, R))$$

and therefore

$$\begin{aligned} \liminf_{R \rightarrow 0} \frac{\log \mu \circ f^{-1}(B_Y(\tilde{y}, R))}{\log R} &= \liminf_{R \rightarrow 0} \frac{\log \mu(f^{-1}(B_Y(\tilde{y}, R)))}{\log R} \geq \limsup_{R \rightarrow 0} \frac{\log \mu(B_X(f^{-1}(\tilde{y}), \frac{1}{K}R))}{\log R} \\ &= \liminf_{R \rightarrow 0} \frac{\log \mu(B_X(f^{-1}(\tilde{y}), \frac{1}{K}R))}{\log(\frac{1}{K}R) + \log K} = d\mathbb{1}_{\text{supp}(\mu)}(f^{-1}(\tilde{y})) = d\mathbb{1}_{\text{supp}(\mu \circ f^{-1})}(\tilde{y}). \end{aligned}$$

Combining these two bounds for the lim sup and the lim inf we obtain

$$\lim_{R \rightarrow 0} \frac{\log \mu \circ f^{-1}(\mathcal{B}_Y(\tilde{y}, R))}{\log R} = d\mathbb{1}_{\text{supp}(\mu \circ f^{-1})}(\tilde{y}),$$

that is the local pointwise dimension of $\mu \circ f^{-1}$ exists and equals the one of μ .

Now to complete the proof of the theorem we just have to notice that the support of the measure μ_ω is almost surely compact, since the support is contained in the global weak set attractor (see [24], Theorem 2), which is compact. The existence of a global weak set attractor for an IOUF is obtained in the next section. Of course this implies that $\mu_\omega \circ g^{-1}$ has a compact support in \mathcal{S}^N . Now apply the statement of the claim ω -wise to obtain that \mathbb{P} -almost surely

$$d(\mu_\omega) = \mathcal{D}(\lambda_1, \dots, \lambda_d),$$

which completes the proof. \square

Theorem 3.5 might be of interest also because one can link the support of the statistical equilibrium to the global set and point attractors, as it is done in [7] and [15].

Theorem 4.3 in [7] states that if φ is a white noise RDS, that is it has independent increments, then the strong global point attractor $A_p^s(\omega)$, if it exists, supports every invariant probability measure, i.e. for the factorization μ_ω of an arbitrary invariant measure it holds

$$\mu_\omega(A_p^s(\omega)) = 1.$$

Clearly we have also $\mu_\omega(A^s(\omega)) = 1$, where $A^s(\omega)$ is the global strong attractor, as in general $A_p^s(\omega) \subset A^s(\omega)$.

Gunter Ochs has shown in [24] (Theorem 2) that $\mu_\omega(A_p(\omega)) = 1$ where $A_p(\omega)$ is the minimal global weak point attractor if it exists. Kuksin and Shirikyan [15] (Proposition 1.6) consider the relations between the weak global point attractor and the support of the random measure μ_ω . They consider RDS with discrete time and independent increments. They have shown that under the conditions C1 and C2 stated below (additional to the assumption of independent increments), the support of the factorization μ_ω obtained via a limiting procedure as in Theorem 3.1 coincides almost surely (μ_ω is itself \mathbb{P} -almost surely defined!) with the unique minimal weak point attractor if it exists. The conditions C1 and C2 are:

C1 The one point motion has unique invariant probability measure ν and the family $(\varphi(t, x))_{t \in \mathbb{R}}$ converges in distribution as $t \rightarrow \infty$ to this invariant probability ν for all x .

C2 For any $x \in \mathbb{R}^d$ and $\epsilon > 0$, there is a $\Omega_\epsilon \in \mathcal{F}$, a compact set $K_\epsilon \subset \mathbb{R}^d$ and an integer k_ϵ , such that $\mathbb{P}(\Omega_\epsilon) \geq 1 - \epsilon$ and

$$\varphi(k, x, \theta_{-k}\omega) \in K_\epsilon \quad \text{for all } \omega \in \Omega_\epsilon \text{ and } k \geq k_\epsilon.$$

Observe that the condition C2 is not automatically satisfied if the RDS has a weak point attractor, since we need the set Ω_ϵ to be independent of k . However, even if we know that the weak point attractor coincides with the support of the statistical equilibrium we still get only a bound from below for its Hausdorff dimension via the dimension of the statistical equilibrium, because the latter provides a bound from below for the Hausdorff dimension of $\text{supp}(\mu_\omega)$ (recall (3.12) and the discussion below).

Proposition 3.6. *Let φ be a RDS on \mathbb{R}^d with time \mathbb{R} over the MDS $(\Omega, \mathcal{F}, (\theta_t)_{t \in \mathbb{R}}, \mathbb{P})$ with global weak point attractor $A_p(\omega)$. Let also φ correspond to some IOUF ϕ with strictly positive Lyapunov exponent in the sense that $\{\phi_{s,t} : s, t \geq 0\}$ and $\{\varphi(t-s, \theta_s \omega) : s, t \geq 0\}$ coincide in distribution. Then*

$$\dim_H(A_p(\omega)) \geq \mathcal{D}(\lambda_1, \dots, \lambda_d),$$

where $\dim_H(A)$ denotes the Hausdorff dimension of A .

Proof:

As mentioned above, the existence of a weak point attractor $A_p(\omega)$ is supplied by Theorem 4.6 and we take (as in [15]) the minimal one. The fact that $\text{supp}(\mu_\omega) \subset A_p(\omega)$ almost surely has been discussed above. The rest is due to the fact that $d(\mu) = \inf \{\dim_H Y : Y \in \mathcal{B}(\mathbb{R}^d), \mu(Y) = 1\}$ (see e.g. [27]). \square

We expect that the dimension of the statistical equilibrium of an IBF can be obtained as the limit of the dimensions of the statistical equilibrium of the corresponding IOUF as $c \rightarrow 0$. In the very special volume preserving case where the Lyapunov exponents of the IBF sum up to zero one easily sees that indeed $\mathcal{D}_c(\lambda_1, \dots, \lambda_d) \rightarrow d$.

4 Weak attractors for isotropic Ornstein-Uhlenbeck flows

4.1 Weak attractors

In this section we give a brief introduction to the notion of a weak random attractor. Recall that the Hausdorff distance $d_H(A, B)$ between two subsets $A, B \subset X$ of a metric space (X, d) is defined as $d_H(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$.

Definition 4.1. A random subset $A(\omega) \subset \mathbb{R}^d$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *random compact set* if

- (a) $A(\omega)$ is nonempty and compact for all $\omega \in \Omega$.
- (b) The mapping $\omega \mapsto \text{dist}(x, A(\omega))$ is measurable for each $x \in \mathbb{R}^d$.

We now give a definition of a global weak attractor following [24].

Definition 4.2. Let φ be a RDS on \mathbb{R}^d over the metric DS $(\Omega, \mathcal{F}, (\theta_t)_{t \in \mathbb{R}}, \mathbb{P})$. The random set $A(\omega)$ is called a *weak global set attractor* for φ if

- (a) $A(\omega)$ is a random compact set.
- (b) A is strictly φ -invariant, that is $\varphi(t, \omega)(A(\omega)) = A(\theta_t \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$.
- (c) $\lim_{t \rightarrow \infty} d_H(\varphi(t, \theta_{-t} \omega)(B), A(\omega)) = 0$ in probability for all compact sets B .

Remark 4.3. Actually the original definition in [24] requires that the weak attractor attracts all random sets in some basin. However, in our special case (global set attractor, state space \mathbb{R}^d) the requirement of attracting all deterministic compact sets already determines the weak set attractor uniquely (see Theorem 3 and Corollaries 3.1 and 3.2 in [24]).

Observe that the random attractor of a RDS gives comprehensive information on the asymptotics of the flow.

If in Definition 4.2 (c) we replace the convergence in probability with almost sure convergence we obtain the *global strong set attractor* as introduced in [9].

Remark 4.4. According to [7](Remark 3.2(ii)) or [6](Section 5) the global strong set attractor is unique. It is already uniquely determined by attracting all compact subsets.

From now on, if not explicitly stated, we will always consider global weak set attractors. One can also define the global weak and strong point attractors:

Definition 4.5. Let φ be a RDS on \mathbb{R}^d over the metric DS $(\Omega, \mathcal{F}, (\theta)_{t \in \mathbb{R}}, \mathbb{P})$. The random set $A_p(\omega)$ is called a *global strong (weak) point attractor* for φ if

- (a) $A_p(\omega)$ is a random compact set.
- (b) A_p is strictly φ -invariant, that is $\varphi(t, \omega)(A_p(\omega)) = A_p(\theta_t \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$.
- (c) $\lim_{t \rightarrow \infty} \text{dist}(\varphi(t, x, \theta_{-t} \omega), A_p(\omega)) = 0$ \mathbb{P} -almost surely (in probability) for all $x \in \mathbb{R}^d$.

As it has been already mentioned in [7] point attractors need not be unique. However it is a direct consequence of Theorem 3.4 in [7] that if a global strong point attractor exists, then there is a unique minimal point attractor. Clearly the global strong set attractor is also a point attractor but not necessarily the minimal one. The same is stated for weak point attractors in [16].

4.2 The case of IOUF

Random attractors are defined for two-sided RDS, however our object of study is a class of stochastic flows. As already mentioned in Remark 2.4 to a stochastic flow ϕ with stationary increments generated via a Kunita type SDE and satisfying some regularity conditions one can canonically construct a corresponding two-sided RDS φ having the same distribution. When we say that the IOUF has a weak attractor we mean that the corresponding RDS has this property. We now state the main result in this section.

Theorem 4.6. *Every isotropic Ornstein-Uhlenbeck flow ϕ has a global weak set attractor.*

The proof of the theorem will rely on the following general criteria for weak attractors, which might be of interest also in a different context. An ongoing joint work of one of the authors with Michael Scheutzow and Hans Crauel [8] deals among others with the following criteria in a very general framework. The proof in the following rather restricted setting can be found in [11].

Theorem 4.7. *Let $\varphi: \mathbb{R} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ be a perfect cocycle on \mathbb{R}^d over the metric dynamical system $(\Omega, \mathcal{F}, (\theta_t)_{t \in \mathbb{R}}, \mathbb{P})$. Then the following are equivalent:*

- (i) *There is a random DS, $\tilde{\varphi}$ over the same MDS, which is indistinguishable from φ (i.e. $\mathbb{P}(\varphi(t) = \tilde{\varphi}(t) \text{ for all } t \in \mathbb{R}) = 1$) and has a global weak set attractor.*

- (ii) For every $\epsilon > 0$ there exists $R_0 > 0$, such that for all $R > 0$ there is a $t_0 > 0$ with the property that for all $t \geq t_0$,

$$\mathbb{P}(\varphi(t)(B(0, R)) \subset B(0, R_0)) \geq 1 - \epsilon.$$

We will actually not verify the sufficient and necessary condition for the existence of a weak random attractor stated above but rather use the following simple corollary of the preceding theorem. The proof is a straightforward application of the Markov inequality.

Corollary 4.8. *Let φ be an RDS on \mathbb{R}^d over the MDS $(\Omega, \mathcal{F}, (\theta_t)_{t \in \mathbb{R}}, \mathbb{P})$. Assume that there is a constant $M > 0$, such that for all $R > 0$ there exists $t_0 := t_0(R) > 0$ with the property that for all $t \geq t_0$*

$$\mathbb{E}[\sup\{|\varphi(t, x)| : x \in B(0, R)\}] \leq M.$$

Then φ has an indistinguishable version $\tilde{\varphi}$ in the sense of the previous theorem, which has a global weak set attractor.

4.3 Proof of the existence of weak attractor

The proof of the main theorem depends crucially on the *chaining technique*. We will use the formulation from [5], where also the proof can be found.

Let (\mathbb{X}, d) be a compact metric space and $\phi: \mathbb{X} \rightarrow \mathbb{R}_+$ be a random continuous function i.e. a random variable taking values in the set of continuous functions from \mathbb{X} to \mathbb{R}_+ . Given a sequence of positive real numbers $(\delta_i)_{i \geq 0}$, such that $\sum_{i=0}^{\infty} \delta_i < \infty$ we determine a sequence $(\chi_i)_{i=0}^{\infty}$ of discretizations (skeletons) of \mathbb{X} , with the property that for all $x \in \mathbb{X}$ there is a point $x_i \in \chi_i$, such that $d(x, x_i) \leq \delta_i$. Assume that $\chi_0 = \{x_0\}$, with $d(x, x_0) \leq \delta_0$ for all $x \in \mathbb{X}$.

Proposition 4.9. (*Chaining*) *Let $\phi: \mathbb{X} \rightarrow \mathbb{R}_+$ be an almost surely continuous random function with $(\delta_i)_{i \geq 0}$ and $(\chi_i)_{i \geq 0}$ as above. For arbitrary positive $\epsilon, z \geq 0$ and an arbitrary sequence of positive reals $(\epsilon_i)_{i \geq 0}$ such that $\epsilon + \sum_{i=0}^{\infty} \epsilon_i = 1$ we have*

$$\mathbb{P}\left(\sup_{x \in \mathbb{X}} \phi(x) > z\right) \leq \mathbb{P}(\phi(x_0) > \epsilon z) + \sum_{i=0}^{\infty} |\chi_{i+1}| \sup_{d(x, y) \leq \delta_i} \mathbb{P}(|\phi(x) - \phi(y)| > \epsilon_i z).$$

Proof. : See [5, Lemma 4.1]. □

The following lemma simply states some well known facts about the running maximum of a standard Brownian motion as well as a common estimate for the Gaussian tails, which we use often below and therefore are stated explicitly.

Lemma 4.10. *Let $(B_t)_{t \geq 0}$ be a standard Brownian motion, and let $B_t^* := \sup_{s \leq t} B_s$ be its running maximum. The distribution of B_t^* has density*

$$\mathbb{1}_{[0, \infty)}(x) \sqrt{\frac{2}{\pi t}} e^{-\frac{x^2}{2t}}$$

with respect to the Lebesgue measure. Moreover for arbitrary $c > 0$ the following bounds hold:

$$\mathbb{P}(B_t \geq c) \leq \frac{1}{c} \sqrt{\frac{t}{2\pi}} e^{-\frac{c^2}{2t}} \quad \text{and} \quad \mathbb{P}(B_t^* \geq c) \leq \frac{1}{c} \sqrt{\frac{2t}{\pi}} e^{-\frac{c^2}{2t}}.$$

The proof of the main result will be splitted into several lemmas.

Lemma 4.11. *For all $\gamma_0 > 0$ there exist $t_0 > 0$ and $R_0 > 0$, such that for all $\gamma \geq \gamma_0$, $t \geq t_0$ and $R \geq R_0$*

$$\mathbb{P}(|\phi_t(x)| > \gamma R) \leq \exp(-k\gamma^2 R^2)$$

holds for all x with $|x| = R$, where k is a strictly positive constant depending on the dimension of the state space d and the slope of the potential c .

Proof. Because of the rotational invariance of the law of ϕ_t we can assume without loss of generality that $x = (R, 0, \dots, 0)^T$. Observe that

$$\begin{aligned} \mathbb{P}(|\phi_t(x)| \geq \gamma R) &= \mathbb{P}\left(\sum_{i=1}^d \phi_t^i(x)^2 \geq \gamma^2 R^2\right) \\ &\leq \mathbb{P}(\phi_t^1(x)^2 \geq \frac{1}{4}\gamma^2 R^2) + \sum_{i=2}^d \mathbb{P}(\phi_t^i(x)^2 \geq \frac{3}{4(d-1)}\gamma^2 R^2) \\ &= \mathbb{P}(|\phi_t^1(x)| \geq \frac{1}{2}\gamma R) + \sum_{i=2}^d \mathbb{P}(|\phi_t^i(x)| \geq \gamma R \sqrt{\frac{3}{4(d-1)}}). \end{aligned}$$

Since $\phi_t(x)$ is an Ornstein-Uhlenbeck process started in $(R, 0, \dots, 0)$, the distributions of $\phi_t(x)^1$ and $\phi_t(x)^i$ for $i = 2, \dots, d$ are given by $\mathcal{N}(Re^{-ct}, \frac{1-e^{-2ct}}{2c})$ and $\mathcal{N}(0, \frac{1-e^{-2ct}}{2c})$ respectively (see [14], section 5.6, page 358). Therefore, with a standard Gaussian random variable B , we have:

$$\begin{aligned} &\mathbb{P}(|\phi_t(x)| \geq \gamma R) \\ &\leq \mathbb{P}\left(\left|\sqrt{\frac{1-e^{-2ct}}{2c}}B + Re^{-ct}\right| \geq \frac{1}{2}\gamma R\right) + \sum_{i=2}^d \mathbb{P}\left(\left|\sqrt{\frac{1-e^{-2ct}}{2c}}B\right| \geq \gamma R \sqrt{\frac{3}{4(d-1)}}\right) \\ &= \mathbb{P}\left(B \notin \left(R\sqrt{\frac{2c}{1-e^{-2ct}}}(-\frac{\gamma}{2} - e^{-ct}), R\sqrt{\frac{2c}{1-e^{-2ct}}}(\frac{\gamma}{2} - e^{-ct})\right)\right) \\ &\quad + \sum_{i=2}^d \mathbb{P}\left(B \notin \left(-\gamma R \sqrt{\frac{3}{4(d-1)}}\sqrt{\frac{2c}{1-e^{-2ct}}}, \gamma R \sqrt{\frac{3}{4(d-1)}}\sqrt{\frac{2c}{1-e^{-2ct}}}\right)\right) \\ &\leq 2\mathbb{P}\left(B \geq R\sqrt{\frac{2c}{1-e^{-2ct}}}(\frac{\gamma}{2} - e^{-ct})\right) + 2\sum_{i=2}^d \mathbb{P}\left(B \geq \gamma R \sqrt{\frac{3}{4(d-1)}}\sqrt{\frac{2c}{1-e^{-2ct}}}\right) \\ &\leq 2\mathbb{P}\left(B \geq R\sqrt{2c}(\frac{\gamma}{2} - e^{-ct})\right) + 2\sum_{i=2}^d \mathbb{P}\left(B \geq \gamma R \sqrt{\frac{6c}{4(d-1)}}\right). \end{aligned}$$

Choose $t_0 > 0$ large enough for $\frac{\gamma}{2} - e^{-ct} \geq \frac{\gamma}{4}$ to hold for all $t \geq t_0$. Thus for all $t \geq t_0$ with the help of

Lemma 4.10 we obtain

$$\begin{aligned} \mathbb{P}(|\phi_t(x)| \geq \gamma R) &\leq 2\mathbb{P}\left(B \geq R\sqrt{2c}\frac{\gamma}{4}\right) + 2\sum_{i=2}^d \mathbb{P}\left(B \geq \gamma R\sqrt{\frac{6c}{4(d-1)}}\right) \\ &\leq 2\frac{1}{\sqrt{2\pi}}\frac{1}{R\sqrt{2c}\frac{\gamma}{4}} \exp\left(-\frac{1}{2}\frac{2c}{16}R^2\gamma^2\right) + 2\frac{d-1}{\sqrt{2\pi}}\frac{1}{\gamma R\sqrt{\frac{6c}{4(d-1)}}} \exp\left(-\frac{1}{2}\frac{6c}{4(d-1)}R^2\gamma^2\right). \end{aligned}$$

Let $R_0 > 0$ be such that

$$\frac{2}{\sqrt{2\pi}}\frac{1}{R_0\sqrt{2c}\frac{\gamma_0}{4}} \leq \frac{1}{2} \quad \text{and} \quad 2\frac{d-1}{\sqrt{2\pi}}\frac{1}{\gamma_0 R_0\sqrt{\frac{6c}{4(d-1)}}} \leq \frac{1}{2}.$$

Then for all $R \geq R_0$ and $\gamma \geq \gamma_0$ (and of course $t \geq t_0$) and with $k = \min\{\frac{c}{16}, \frac{3c}{4(d-1)}\}$ we obtain

$$\mathbb{P}(|\phi_t(x)| \geq \gamma R) \leq \frac{1}{2} \exp\left(-\frac{c}{16}R^2\gamma^2\right) + \frac{1}{2} \exp\left(-\frac{1}{2}\frac{6c}{4(d-1)}R^2\gamma^2\right) \leq 2 \exp\left(-k\gamma^2 R^2\right),$$

which proves the statement. \square

The following lemma uses the chaining technique and is crucial for the proof of the main theorem.

Lemma 4.12. *For every $t > 0$ there exist constants $K > 0$ and $L > 0$ (depending on $t!$), such that for all $R > 0$*

$$\mathbb{E}\left[\sup_{s \in [0, t]} \left\{ \sup_{x \in B(0, R)} |\phi_s(x)| \right\}\right] \leq KR + L.$$

Proof. Observe the simple computation:

$$\begin{aligned} \mathbb{E}\left[\sup_{s \in [0, t]} \left\{ \sup_{x \in B(0, R)} |\phi_s(x)| \right\}\right] &\leq \mathbb{E}\left[\sup_{s \in [0, t]} \left\{ \sup_{x \in B(0, R)} |\phi_s(x) - \phi_s(0)| \right\}\right] \\ &+ \mathbb{E}\left[\sup_{s \in [0, t]} |\phi_s(0)|\right] = \int_0^\infty \mathbb{P}\left[\sup_{s \in [0, t]} \left\{ \sup_{x \in B(0, R)} |\phi_s(x) - \phi_s(0)| \right\} \geq z\right] dz + \mathbb{E}\left[\sup_{s \in [0, t]} |\phi_s(0)|\right]. \end{aligned}$$

As $(\phi_t(x))$ is an Ornstein-Uhlenbeck process we certainly have a constant L depending on t such that $\mathbb{E}\left[\sup_{s \in [0, t]} |\phi_s(0)|\right] \leq L$.

In the following we show the remaining part of Lemma 4.12, i.e. that there exists $K > 0$, such that for all $R > 0$

$$\int_0^\infty \mathbb{P}\left[\sup_{s \in [0, t]} \left\{ \sup_{x \in B(0, R)} |\phi_s(x) - \phi_s(0)| \right\} > z\right] dz \leq KR.$$

Applying the chaining technique to the function $\sup_{s \in [0, t]} |\phi_s(x) - \phi_s(0)|$ with suitable χ ,

$$\mathbb{X} = B(0, R), \quad 0 < \epsilon < 1, \quad \epsilon_i = (1 - \epsilon)\frac{e - 1}{e}e^{-i} \quad \text{and} \quad \delta_i = Re^{-2i}$$

we get for arbitrary $m > 0$ and some $c_1 > 0$:

$$\begin{aligned}
& \mathbb{P}[\sup\{\sup_{s \in [0, t]} |\phi_s(x) - \phi_s(0)| : x \in B(0, R)\} > mR] \leq \mathbb{P}[\sup_{s \in [0, t]} |\phi_s(0) - \phi_s(0)| > \epsilon mR] \\
& + \sum_{i=0}^{\infty} |\chi_{i+1}| \sup_{|x-y| \leq \delta_i} \mathbb{P}[\sup_{s \in [0, t]} |\phi_s(x) - \phi_s(0)| - \sup_{s \in [0, t]} |\phi_s(y) - \phi_s(0)| > \epsilon_i mR] \\
& \leq c_1 \sum_{i=0}^{\infty} \left(\frac{R}{Re^{-2(i+1)}} \right)^d \sup_{|x-y| \leq \delta_i} \mathbb{P}[\sup_{s \in [0, t]} \{|\phi_s(x) - \phi_s(0)| - |\phi_s(y) - \phi_s(0)|\} > \epsilon_i mR] \\
& \leq c_1 \sum_{i=0}^{\infty} e^{2d(i+1)} \sup_{|x-y| \leq \delta_i} \mathbb{P}[\sup_{s \in [0, t]} |\phi_s(x) - \phi_s(y)| > \epsilon_i mR].
\end{aligned}$$

According to Corollary 2.3 there are constants $\lambda, \sigma > 0$, such that for some standard Brownian motion $(B_t)_{t \geq 0}$ it holds almost surely that

$$\sup_{s \in [0, t]} |\phi_s(x) - \phi_s(y)| \leq |x - y| \exp(\sigma B_t^* + \lambda t).$$

Therefore we have

$$\begin{aligned}
& \mathbb{P}[\sup\{\sup_{s \in [0, t]} |\phi_s(x) - \phi_s(0)| : x \in B(0, R)\} > mR] \\
& \leq c_1 \sum_{i=0}^{\infty} e^{2d(i+1)} \mathbb{P}[\delta_i \exp(\sigma B_t^* + \lambda t) > \epsilon_i mR] \leq c_1 \sum_{i=0}^{\infty} e^{2d(i+1)} \mathbb{P}[B_1^* > \frac{1}{\sigma \sqrt{t}} \ln \frac{\epsilon_i mR}{\delta_i} - \frac{\lambda}{\sigma} \sqrt{t}] \\
& = c_1 \sum_{i=0}^{\infty} e^{2d(i+1)} \mathbb{P}[B_1^* > \frac{1}{\sigma \sqrt{t}} (\ln \frac{(1-\epsilon)(e-1)}{e} + \ln m + i - \lambda t)].
\end{aligned}$$

Let $m_0 > 0$ be so large that

$$\ln \frac{(1-\epsilon)(e-1)}{e} + \ln m_0 - \lambda t > \sigma \sqrt{t}.$$

Then for all $m > m_0$ with the help of Lemma 4.10 we get for some $c_2 > 0$

$$\begin{aligned}
& \mathbb{P}[\sup\{\sup_{s \in [0, t]} |\phi_s(x) - \phi_s(0)| : x \in B(0, R)\} > mR] \\
& \leq c_2 \sum_{i=0}^{\infty} \exp(2d(i+1) - \frac{1}{2\sigma^2 t} (\ln \frac{(1-\epsilon)(e-1)}{e} + \ln m + i - \lambda t)^2) =: c_2 \sum_{i=0}^{\infty} a_i,
\end{aligned}$$

with

$$a_i = \exp(2d(i+1) - \frac{1}{2\sigma^2 t} (\ln \frac{(1-\epsilon)(e-1)}{e} + \ln m + i - \lambda t)^2).$$

Choose $m_1(\lambda, \sigma, t, \epsilon) = m_1 > m_0$ so big that for all $m > m_1$ the inequality

$$\frac{a_{i+1}}{a_i} = \exp(2d - \frac{1}{\sigma^2 t} (\ln m + i + \frac{1}{2} - \lambda t + \ln \frac{(1-\epsilon)(e-1)}{e})) \leq e^{-1}$$

holds for all $i \in \mathbb{N} \cup \{0\}$ (that is, the inequality holds for $i = 0$). Then for all $m > m_1$ we have

$$\begin{aligned} & \mathbb{P}[\sup\{\sup_{s \in [0, t]} |\phi_s(x) - \phi_s(0)| : x \in B(0, R)\} > mR] \\ & \leq c_2 \sum_{i=0}^{\infty} a_i \leq c_2 a_0 \sum_{i=0}^{\infty} e^{-i} = c_2 \frac{e}{e-1} \exp\left(2d - \frac{1}{2\sigma^2 t} \left(\ln \frac{(1-\epsilon)(e-1)}{e} + \ln m - \lambda t\right)^2\right) \end{aligned}$$

and clearly there are positive constants $p(t, \sigma, \lambda, d)$ and $q(t, \sigma, \lambda, d)$, such that for all $m \geq m_1$

$$\mathbb{P}[\sup\{\sup_{s \in [0, t]} |\phi_s(x) - \phi_s(0)| : x \in B(0, R)\} > mR] \leq p \exp(-q \ln^2 m). \quad (4.15)$$

Using the last inequality we finally get

$$\begin{aligned} & \mathbb{E}[\sup\{\sup_{s \in [0, t]} |\phi_s(x) - \phi_s(0)| : x \in B(0, R)\}] \\ & = R \int_0^{\infty} \mathbb{P}[\sup\{\sup_{s \in [0, t]} |\phi_s(x) - \phi_s(0)| : x \in B(0, R)\} \geq mR] dm \\ & \leq Rm_1 + R \int_{m_1}^{\infty} \mathbb{P}[\sup\{\sup_{s \in [0, t]} |\phi_s(x) - \phi_s(0)| : x \in B(0, R)\} \geq mR] dm \\ & \leq Rm_1 + R \int_{m_1}^{\infty} p \exp(-q \ln^2 m) dm \leq KR \end{aligned}$$

for some $K > 0$. □

The following corollary is a consequence of the preceding two lemmata

Corollary 4.13.

- (i) *For every $t > 0$ there exists $R_0 > 0$ (depending on t), such that for all $R \geq R_0$ and all $x \in \mathbb{R}^d$*

$$\mathbb{P}(\text{diam}(\phi_t(B(x, 1))) \geq R) \leq c_1(t, \sigma, \lambda, d) \exp(-c_2(t, \sigma, \lambda, d) \ln^2 R) \quad (4.16)$$

holds for all $R \geq R_0$ with some positive constants $c_1(t, \sigma, \lambda, d)$ and $c_2(t, \sigma, \lambda, d)$ (depending on t, σ, λ and d).

- (ii) *For all $\gamma_0 > 0$ there exist $t_0 > 0$ and $R_0 > 0$ such that for all $\gamma \geq \gamma_0$ and $R \geq R_0$*

$$\mathbb{P}(\sup\{|\phi_{t_0}(x)| : x \in B(0, R)\} > \gamma R) \leq c_1 \exp(-c_2 \ln^2(\gamma R)), \quad (4.17)$$

where c_1 and c_2 are positive constants depending on d, λ, σ, c and t_0 .

Proof. (i) follows immediately from (4.15) (ii) is a combination of (i) and Lemma 4.11. □

Proposition 4.14. *For all $\delta \in (0, 1)$ there exist $t_0 > 0$ and $R_0 > 0$, such that for all $R > 0$*

$$\mathbb{E}[\sup\{|\phi_{t_0}(x)| : x \in B(0, R)\}] \leq \delta R \mathbb{1}_{\{R \geq R_0\}} + \delta R_0 \mathbb{1}_{\{R < R_0\}}.$$

Remark 4.15. For our purposes the existence of one such δ is enough.

Proof.

Step 1: Choose arbitrary $\delta \in (0, 1)$. We first show that there exist $t_0 > 0$ and $R_0 > 0$, such that for all $R \geq R_0$

$$\mathbb{E} [\sup\{|\phi_{t_0}(x)| : x \in B(0, R)\}] \leq \delta R.$$

Let $\gamma_0 = \frac{\delta}{2}$. Choose $R_{0,1} > 0$ and $t_0 > 0$ according to (4.17). Thus for all $R \geq R_{0,1}$

$$\begin{aligned} \mathbb{E} [\sup\{|\phi_{t_0}(x)| : x \in B(0, R)\}] &= R \int_0^\infty \mathbb{P}(\sup\{|\phi_{t_0}(x)| : x \in B(0, R)\} > \gamma R) d\gamma \\ &\leq R\gamma_0 + R \int_{\gamma_0}^\infty \mathbb{P}(\sup\{|\phi_{t_0}(x)| : x \in B(0, R)\} > \gamma R) d\gamma \leq R\gamma_0 + R \int_{\gamma_0}^\infty c_1 \exp(-c_2 \ln^2(\gamma R)) d\gamma, \end{aligned}$$

where c_1 and c_2 are as in (4.17). Choose $R_{0,2} > 0$ such that

$$\int_{\gamma_0}^\infty c_1 \exp(-c_2 \ln^2(\gamma R)) ds \leq \gamma_0,$$

which is possible as the whole expression on the left-hand side goes to zero with $R \rightarrow +\infty$. Clearly, for all $R > R_0 := \max\{R_{0,1}, R_{0,2}\}$ it holds that $\mathbb{E} [\sup\{|\phi_{t_0}(x)| : x \in B(0, R)\}] \leq \delta R$.

Step 2: For arbitrary $R > 0$, using the result of Step 1 we have

$$\begin{aligned} \mathbb{E} [\sup\{|\phi_{t_0}(x)| : x \in B(0, R)\}] &= \mathbb{1}_{\{R \geq R_0\}} \mathbb{E} [\sup\{|\phi_{t_0}(x)| : x \in B(0, R)\}] + \mathbb{1}_{\{R < R_0\}} \mathbb{E} [\sup\{|\phi_{t_0}(x)| : x \in B(0, R)\}] \\ &\leq \mathbb{1}_{\{R \geq R_0\}} \delta R + \mathbb{1}_{\{R < R_0\}} \mathbb{E} [\sup\{|\phi_{t_0}(x)| : x \in B(0, R_0)\}] \leq \mathbb{1}_{\{R \geq R_0\}} \delta R + \mathbb{1}_{\{R < R_0\}} \delta R_0, \end{aligned}$$

where R_0 is of course as in Step 1. □

Now we give the proof of the main result:

Proof of the main theorem.

In the following we verify the sufficient condition for the existence of a random weak attractor stated in Corollary 4.8, i.e. for an IOUF ϕ we show that there exists a constant $M > 0$, such that for every $R > 0$ there exists a $T > 0$ with

$$\mathbb{E} [\sup\{|\phi_t(x)| : x \in B(0, R)\}] \leq M \quad \text{for all } t \geq T.$$

We do this in two steps:

Step 1: Choose arbitrary $\delta \in (0, 1)$. Take $t_0 > 0$ and $R_0 > 0$ as in Proposition 4.14, that is for all $R > 0$,

$$\mathbb{E} [\sup\{|\phi_{t_0}(x)| : x \in B(0, R)\}] \leq \delta R \mathbb{1}_{\{R \geq R_0\}} + \delta R_0 \mathbb{1}_{\{R < R_0\}}.$$

We first show that there exists a constant $M_1 > 0$, such that for every $R > 0$ there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\mathbb{E} [\sup\{|\phi_{nt_0}(x)| : x \in B(0, R)\}] \leq M_1 \text{ for all } n \in \mathbb{N}.$$

Observe that for arbitrary $n \geq 1$

$$\begin{aligned} & \mathbb{E}[\sup\{|\phi_{nt_0}(x)| : x \in B(0, R)\} \mid \mathcal{F}_{(n-1)t_0}] \\ &= \mathbb{E}[\sup\{|\phi_{(n-1)t_0, nt_0}(x)| : x \in \phi_{(n-1)t_0}(B(0, R))\} \mid \mathcal{F}_{(n-1)t_0}] = (\star) \end{aligned}$$

and since $\phi_{(n-1)t_0}$ is $\mathcal{F}_{(n-1)t_0}$ -measurable and $\phi_{(n-1)t_0, nt_0}$ is independent of $\mathcal{F}_{(n-1)t_0}$ we obtain

$$\begin{aligned} (\star) &= \mathbb{E}[\sup\{|\phi_{(n-1)t_0, nt_0}(x)| : x \in B\} \mid_{B=\phi_{(n-1)t_0}(B(0, R))}] \\ &\leq \mathbb{E}[\sup\{|\phi_{(n-1)t_0, nt_0}(x)| : x \in B(0, \tilde{R})\} \mid_{\tilde{R}=\sup\{|\phi_{(n-1)t_0}(x)| : x \in B(0, R)\}}] \\ &= \mathbb{E}[\sup\{|\phi_{t_0}(x)| : x \in B(0, \tilde{R})\} \mid_{\tilde{R}=\sup\{|\phi_{(n-1)t_0}(x)| : x \in B(0, R)\}}] \end{aligned}$$

where the last equality is due to the time homogeneity of the flow. Applying Proposition 4.14 we get

$$\begin{aligned} & \mathbb{E}[\sup\{|\phi_{nt_0}(x)| : x \in B(0, R)\} \mid \mathcal{F}_{(n-1)t_0}] \\ &\leq \mathbb{E}[\sup\{|\phi_{t_0}(x)| : x \in B(0, \tilde{R})\} \mid_{\tilde{R}=\sup\{|\phi_{(n-1)t_0}(x)| : x \in B(0, R)\}}] \\ &\leq \left[\delta \tilde{R} \mathbb{1}_{\{\tilde{R} \geq R_0\}} + \delta R_0 \mathbb{1}_{\{\tilde{R} < R_0\}} \right]_{\tilde{R}=\sup\{|\phi_{(n-1)t_0}(x)| : x \in B(0, R)\}} \end{aligned}$$

That is, with $X_n^R := \sup\{|\phi_{nt_0}(x)| : x \in B(0, R)\}$ and $\mathcal{G}_n := \mathcal{F}_{nt_0}$ we have

$$\mathbb{E}[X_n^R \mid \mathcal{G}_{n-1}] \leq \delta X_{n-1}^R \mathbb{1}_{\{X_{n-1}^R \geq R_0\}} + \delta R_0 \mathbb{1}_{\{X_{n-1}^R < R_0\}}. \quad (4.18)$$

Now the following iteration argument:

$$\begin{aligned} \mathbb{E}[X_n^R] &= \mathbb{E}[\dots \mathbb{E}[X_n^R \mid \mathcal{G}_{n-1}] \mid \mathcal{G}_{n-2}] \dots \mid \mathcal{G}_0] \\ &\leq \mathbb{E}[\dots \mathbb{E}[\delta X_{n-1}^R \mathbb{1}_{\{X_{n-1}^R \geq R_0\}} + \delta R_0 \mathbb{1}_{\{X_{n-1}^R < R_0\}} \mid \mathcal{G}_{n-2}] \dots \mid \mathcal{G}_0] \\ &\leq \delta \mathbb{E}[\dots \mathbb{E}[X_{n-1}^R \mid \mathcal{G}_{n-2}] \dots \mid \mathcal{G}_0] + \delta R_0 \leq \delta^2 \mathbb{E}[\dots \mathbb{E}[X_{n-2}^R \mid \mathcal{G}_{n-3}] \dots \mid \mathcal{G}_0] + \delta^2 R_0 + \delta R_0 \leq \dots \\ &\leq \delta^{n-1} \mathbb{E}[X_1^R] + R_0 \delta (1 + \delta + \dots + \delta^{n-1}) \\ &\leq \delta^n R \mathbb{1}_{\{R \geq R_0\}} + \delta^n R_0 \mathbb{1}_{\{R < R_0\}} + R_0 \frac{\delta}{1 - \delta} \leq \delta^n \max\{R, R_0\} + R_0 \frac{\delta}{1 - \delta} \end{aligned}$$

proves the statement, since obviously $\limsup_{n \rightarrow \infty} \mathbb{E}[X_n^R] \leq \frac{R_0 \delta}{1 - \delta}$, and this estimate is independent of R .

Step 2: Let $t = nt_0 + s$ with $s < t_0$ for t_0 as in Step 1.

An analogous argument as in Step 1 together with Lemma 4.12 implies:

$$\begin{aligned}
\mathbb{E}[\sup\{|\phi_t(x)| : x \in B(0, R)\}] &= \mathbb{E}[\mathbb{E}[\sup\{|\phi_t(x)| : x \in B(0, R)\} \mid \mathcal{F}_{nt_0}]] \\
&= \mathbb{E}[\mathbb{E}[\sup\{|\phi_{nt_0,t}(x)| : x \in \phi_{nt_0}(B(0, R))\} \mid \mathcal{F}_{nt_0}]] \\
&\leq \mathbb{E}\left[\mathbb{E}[\sup\{|\phi_s(x)| : x \in B(0, \tilde{R})\}] \Big|_{\tilde{R}=\sup\{|\phi_{nt_0}(x)| : x \in B(0, R)\}}\right] \\
&\leq \mathbb{E}\left[\mathbb{E}[\sup\{\sup_{0 \leq u \leq t_0} |\phi_u(x)| : x \in B(0, \tilde{R})\}] \Big|_{\tilde{R}=\sup\{|\phi_{nt_0}(x)| : x \in B(0, R)\}}\right] \\
&\leq \mathbb{E}[K(t_0) \sup\{|\phi_{nt_0}(x)| : x \in B(0, R)\} + L(t_0)] \quad (\text{Lemma 4.12}) \\
&= K(t_0) \mathbb{E}[\sup\{|\phi_{nt_0}(x)| : x \in B(0, R)\}] + L(t_0) \\
&\leq K(t_0) \delta^n \max\{R, R_0\} + K(t_0) R_0 \frac{\delta}{1 - \delta} + L(t_0) \quad (\text{Step 1})
\end{aligned}$$

and therefore for all $R > 0$

$$\limsup_{t \rightarrow +\infty} \mathbb{E}[\sup\{|\phi_t(x)| : x \in B(0, R)\}] \leq K(t_0) R_0 \frac{\delta}{1 - \delta} + L(t_0).$$

The statement is now proven since the bound above is independent of R . \square

4.4 The case of negative Lyapunov exponent

Our aim here is to show that in the case of negative top Lyapunov exponent the global weak set attractor of an isotropic Ornstein-Uhlenbeck flow is trivial, i.e. contains almost surely only one point. We sketch the proof under the following additional "squeezing" condition.

Condition 4.16. (Squeezing)

We say a stochastic flow ϕ satisfies the squeezing condition if for every $r > 0$ there exists $t_r > 0$, such that the ball with radius r and centered at zero is uniformly squeezed by ϕ in the sense that there exists $\epsilon > 0$ with the property that

$$\mathbb{P}(\phi_{0,t_r} \text{Ball}(0, r + \epsilon) \subset \text{Ball}(0, r - \epsilon)) > 0.$$

The squeezing condition can be verified for IBFs whose potential measure in the spectral decomposition of the corresponding field F is not supported on the set of zeros of certain Bessel functions. Moreover, an ongoing joint work of one of the authors with Steffen Dereich on a support theorem for stochastic flows shows that the squeezing condition is always verified by an IOUF.

Theorem 4.17. *Let ϕ be an isotropic Ornstein-Uhlenbeck flow with generating field $F(t, x) - cxt$ and strictly negative top Lyapunov exponent verifying the squeezing condition. Then the weak attractor $A(\omega)$ of ϕ is trivial, i.e. contains almost surely only one point.*

Proof. We will divide the proof into several steps.

Step 1: Simple iteration of the squeezing condition and the time homogeneity of the IOUF imply that for all $R > r > 0$ there exists $T_\phi^{R,r} > 0$, such that

$$\mathbb{P}(\phi_{nT_\phi^{R,r}, (n+1)T_\phi^{R,r}} \text{Ball}(0, R) \subset \text{Ball}(0, r)) > 0 \quad \text{for all } n \geq 0 \quad (4.19)$$

Step 2: Here we pass to a certain two-sided RDS ψ , corresponding to the isotropic Ornstein-Uhlenbeck flow seen from the point of view of a moving particle.

Consider the semimartingale field $(W(t, x) : t \in \mathbb{R}_+, x \in \mathbb{R}^d)$ defined as the $C^{3,\delta}$ -valued modification (for arbitrary $\delta < 1$) of the field

$$\int_0^t V(du, x + \phi_{0,u}(0)) - \int_0^t V(du, \phi_{0,u}(0)) = -c \int_0^t x du + \int_0^t F(du, x + \phi_{0,u}(0)) - \int_0^t F(du, \phi_{0,u}(0)).$$

The joint variation process is given by $(b_{i,j}(x - y) + b_{i,j}(0) - b_{i,j}(x) - b_{i,j}(y))t$ and since it is deterministic the martingale part of $\{W(t, x) : t \in \mathbb{R}_+, x \in \mathbb{R}^d\}$ is a Gaussian martingale field. Moreover, W has stationary and independent increments. Let ψ be the forward Brownian stochastic flow generated by $W(x, t)$ via the SDE:

$$\psi_{s,t}(x) - x = \int_s^t W(du, \psi_{s,u}(x)) \quad \text{for all } 0 \leq s \leq t \text{ and } x \in \mathbb{R}^d. \quad (4.20)$$

With the help of Theorem 3.3.3 in [17] and the pathwise uniqueness of the solutions of the SDE above we can show that actually

$$\psi_{s,t}(x) = \phi_{s,t}(x + \phi_{0,s}(0)) - \phi_{0,t}(0), \quad (4.21)$$

i.e. ψ is simply the flow ϕ , observed from the point of view of the moving particle started at the origin. Of course the last equality is in the sense that the flows on the left and the right hand side are modifications of each other.

As in [2] we can construct a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_s^t)_{-\infty < s \leq t < +\infty}, \tilde{\mathbb{P}})$, an ergodic filtered metric dynamical system $(\theta_t)_{t \in \mathbb{R}}$ and a perfect cocycle $\psi : \mathbb{R} \times \mathbb{R}^d \times \tilde{\Omega} \rightarrow \mathbb{R}^d$ (we use the same symbol ψ for the RDS and the flow!), such that the distribution of $\{\psi(t - s, \theta_s \tilde{\omega}) : s, t \geq 0\}$ coincides with the distribution of $\{\psi_{s,t} : s, t \geq 0\}$.

It is also easily concluded that for arbitrary $R > r > 0$ there is a $T^{R,r} > 0$, such that with positive probability the ball with Radius R will be squeezed after time $T^{R,r}$ by the action of the cocycle ψ into the ball with radius r , i.e.

$$\tilde{\mathbb{P}}(\psi(T^{R,r}, \tilde{\omega})(B(0, R)) \subset B(0, r)) > 0. \quad (4.22)$$

Step 3: Here we sketch the proof of the fact that the global stable manifold \mathcal{S}_g of the RDS ψ contains with positive probability a ball centered at zero and having arbitrarily large radius. The idea is simple: the distance (d_t) between the stationary (here fixed) trajectory and the complement of the global stable manifold (see [23]) is stationary. However, according to Step 1, the image of arbitrary big ball becomes arbitrary small with strictly positive probability in finite time t_0 , that is with positive probability at t_0 the image of the ball is contained in the

global stable manifold, hence the stationary process (d_t) assumes arbitrarily large values with strict positive probability.

Step 4: The RDS ψ has a weak attractor, since the bound in Corollary 4.8 holds also for ψ if it holds for ϕ . Denote by $A(\tilde{\omega})$ the attractor of the RDS ψ . Similar argument as above implies that $A(\tilde{\omega})$ is contained in the global stable manifold and thus is trivial. The same must hold for the random attractor of ϕ as it is a random shift of the attractor of ψ (see (4.21)) and clearly a random translation of a set containing only one random point is still a random one-point set. . \square

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